Category Theory

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0.1 Introduction

0.1.1 Project Introduction

The problem that this project attempts to address emerges out of the conflict between two simple facts. The first is that the ubiquitous notion of adjunction is fundamental to much modern mathematics, the second is that the notion of adjunction requires a lot of work to even state, let alone understand. So where does this leave the undergraduate student of mathematics? It would be a great boon for them to have the process of adjunction as a conceptual tool, however the work that is required can be prohibitive. Therefore the aims of my project are as follows:

• To explore the idea that adjoint functors, and hence adjointness is pervasive in modern mathematics.

• To provide a path towards understanding adjointness which is as easy for the undergraduate mathematician to understand as possible.

• To give a grounding in the basics topics of Category Theory along the way.

These objectives provide the rational for my presentation of the material. Most works on Category Theory begin by defining a category, a functor, and then natural transformations, before going back and developing notions such as limits and universal constructions. The first half of my project, following the presentation used in Topoi (Goldblatt), extensively develops 'first order' category theory. Goldblatt provides an expansive exploration of morphisms, duality and constructions within and on categories before functors are even introduced. The main reason for my following of this idea is that I feel it provides the easiest introduction to this very abstract branch of mathematics.

In a large part the student of mathematics spends their time working on what can be considered one level of abstraction, there are mathematical objects; functions, numbers, sets, and such, and then there are theories that deal with these objects; such as group theory or topology. A category however represents a whole new level above this (of course these there isn't fundamentally any real demarcation, but it is a useful way of thinking sometimes). To introduce functors and natural transformations immediately is to expect the student to be comfortable not just one step above their normal level of discourse, but also comfortable two and three levels above it. Therefore, I have waited until I feel that a student would have grown sufficiently familiar with the notion of a category before introducing functors and natural transformations.

There is however one difficulty that I have come up against due to my stated aims, namely, how does one show that an idea arises everywhere in mathematics when one has also committed one's self to keeping the presentation as straight forward as possible. The first would seem to warrant an extensive development of many areas of mathematics, the second a choice of one area that provides the easiest path towards the end goal. I have attempted to reach a compromise, whilst I have favoured algebraic examples over more topological or geometric ones, I have also endeavored to include basic examples from other areas. The reason for favouring algebraic examples is that in my experience students (including myself) are often more comfortable with algebra than with geometry and topology.

0.1.2 Introduction to Category Theory

Category theory means many things to many different people, and perhaps due to this, there isn't a single agreed upon characterization that is satisfactory to everyone. At its lowest level category theory is a unifying language spanning all of mathematics and logic, one which, for example, allows connections between topological spaces and abelian groups to be made explicit and rigorous. In fact this was the original purpose for which the notion of a category was formalized by Eilenberg and Mac Lane in the 1940's.

However, the ideas where soon seen to have a life of their own. Firstly, the original notions found application in a huge number of different branches, including Computer Science, Logic, Algebraic Geometry, and Mathematical Physics. Secondly the ideas themselves were found to be sufficiently powerful and flexible to be able to provide an alternative foundation to Mathematics than that given by Set Theory.

0.1.3 Section Summaries

Section 2.0 - Basic Category Theory : Definition of a category, examples of categories - sets, groups, monoids, topological spaces, and posets. Then a look at the relation between monoids and categories; posets and categories. Introduction of diagrams, then the presentation of a few finite categories in terms of diagrams. Finally, a look at small, large and locally small categories before defining a subcategory.

Section 2.1 - Constructing New Categories: The product category, the arrow category, the slice category, the discrete category.

Section 2.2 - Morphisms: Monomorphisms and epimorphisms, monos and epics in $\mathbb{SET}$ equivalent to injectivity and surjectivity, morphisms in $\mathbb{POS}$, isomorphisms in a category, isomorphism $\implies$ monomorphism and epimorphism, but mono and epic does not $\implies$ isomorphism. Isomorphic objects in a category.

Section 2.3 - Duality: The opposite category, dual statements, epimorphisms and monomorphism are dual.

Section 2.4 - Elementary Structures Within Categories: Initial object, initial objects unique up to isomorphism, examples of intial objects, terminal objects, terminal objects unique, examples of terminal objects, zero objects, zero morphisms, examples of zero objects. Product in a category,

Section 3.1 - Functors: Definition of a functor, a couple of examples, functors between monoids, functors between posets, composition of functors, the category of small categories, contravariant functors, hom functors, bifunctors, bifunctors determined by their restriction to one variable.

Section 3.2 - Natural Transformations: The definition of a natural transformation, then the identity natural transformation, natural transformations between posets, natural transformations on the left product fuinctor, natural isomorphisms, composing natural transformations - horizontally vertically and with functors - the functor category, the categories C^1 and C^2, equivalence of categories.

Section 3.3 - Limits: Diagram, cone, limit, uniqueness of limits, examples of limits - product, initial objects, limit theorem,

Section 3.4 - Yoneda's lemma: Review of the hom functors, yoneda lemma, yoneda embedding, then a corollary.

Section 4.1 - Free Objects: free monoid, free functor, rationals, field of fractions

Section 4.2 - Adjoint Functors: first definition of adjunction in terms of unit, free monoid as an adjunction, hom set definition of adjunction, third definition in terms of counits, examples of adjoint functors - initial objects, product, limit, galois connection.

Section 3.3 - The Adjoint Functor Theorem: Adjoint unique up to isomorphism, right adjoint preserve limits, left adjoints preserve co-limits, lemma on solution set criterion, Freyds adjoint functor theorem,

Section 5.1 - Monads: definition of a monad, monad of monoids, algebra for a monad, fourth definition of adjunction in terms of triangle identities, monads from adjunctions, Eilenberg-Moore category, adjunction from this category, kleisli category, adjunction from this category.

Section 5.2 - Monads and Adjoints:

0.2 Basic Category Theory

We begin by giving the definition of a category.

Definition 0.2.1. *A category $\mathcal{C}$ comprises:*
• *A collection of objects, denoted, Obj $\mathcal{C}$.*
• *A collection of morphisms, denoted, Mor $\mathcal{C}$.*
• *Two operations, domain and codomain, that for each $f \in Mor\ C$, assign objects; $dom(f)$, $cod(f)$, called the domain and codomain of f, if we have $A = dom(f)$, $B = cod(f)$, then we write, $f : A \to B$.*
• *For each pair of morphisms, f, g, such that, $f : A \to B$ and $g : B \to C$, there exists a morphism $g \circ f : A \to C$.*
• *For each $A \in Obj\ \mathcal{C}$ there exists a morphism $id_A : A \to A$, such that for all $f : A \to B$ and $g : C \to A$, $f \circ id_A = f$ and $id_A \circ g = g$.*
• *For all $f : A \to B$, $g : B \to C$, $h : C \to D$; $(h \circ g) \circ f = h \circ (g \circ f)$. This is the associativity requirement.*

▷

Remark 0.2.2. • For those unfamiliar with Set Theory, a formal distinction is made between 'sets' and 'classes', basically a class is something that is allowed to be bigger than a set, of you are looking for a quick informal way of thinking about it then: $finite\ set < set < class$. A rule of thumb is you are not allowed to talk about the 'set of all sets', but must instead say the 'class of all sets'.
• In our definition of a category instead of requiring that we have a set of objects, we used the term 'collection' this is because we will often want to talk about aggregates larger than sets (or even classes).
• It would be a mistake to get too stuck on the notion of the morphism as a function, 'acting' on the objects, at the lowest level it merely provides some notion of 'relation' between that domain and codomain.

Example 0.2.3. The category of sets and set functions, henceforth denoted $\mathbb{SET}$.

This is our first example of a category, and a source of much motivation. We will go on to generalize such diverse notions as injectivity, the cartesian product of sets, the empty set, and equivalence relations.

• *Obj* $\mathcal{C}$ is simply taken to be the class of all sets. Due to Russell's paradox, the class of all sets is not itself a set, hence explaining the more lax requirement on a category.
• Mor $\mathcal{C}$ is taken to be the class of all set functions.
• For a given morphism (i.e. a function), the domain and codomain of the morphism are taken to be the standard domain and codomain of the function.
• If we have two functions; $f : A \to B$, $g : B \to C$ then $g \circ f : A \to C$ is also a function when defined by:

$$(g \circ f)(a) = g(f(a)), \forall a \in A.$$

Hence we have composition of morphisms.
• The identity function on a set can be seen to fulfill the role of the identity morphism. Let $f : A \to B$, and $g : C \to A$, then:

$$f \circ id_A(a) = f(id_A(a)) = f(a).$$
$$id_A \circ g(a) = id_A(g(a)) = g(a).$$

• Finally, function composition is known to be associative, giving us the final requirement.

We now give a brief recap of the basic definitions of group theory.

Definition 0.2.4. *A group is a triple,* $(G, \cdot, e_G)$*, where* G *is a set,* $\cdot$ *is a binary operation and* e_G *is the identity element, such that:*
• *For all* $a \in G$*, there exists some* a^{-1} *such that:*

$$a \cdot a_1 = e_G = a^{-1} \cdot a.$$

• *The binary operation* $\cdot$ *is associative, that is for all* g_1*,* g_2*,* g_3*:*

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

▷

Definition 0.2.5. *A group homomorphism* f *from a group* $(G, \cdot, e_G)$ *to a group* $(H, *, e_H)$*, is a function* $f : G \to H$*, such that:*
• $f(e_G) = e_H$
• $f(g_1 \cdot g_2) = f(g_1) * f(g_2)$*, for all* g_1*,* g_2 *in* G*.*

▷

Proposition 0.2.6. *Given two group homomorphisms,* $f : G \to H$ *and* $g : H \to J$*, the composition,* $g \circ f$*, defined by* $(g \circ f)(a) = g(f(a))$*, is a group homomorphism from* G *to* J*.*

Proof. We first check that $g \circ f$ sends identities to identities:

$$(g \circ f)(e_G) = g(f(e_G)) = g(e_H) = e_J.$$

We now need to check that $y \circ x$ respects the binary operation.

$$(g \circ f)(a_1 a_2) = g(f(a_1 a_2)) = g(f(a_1)f(a_2)) = g(f(a_1))g(f(a_2)) = (g \circ f)(a_1)(g \circ f)(a_2).$$

□

Example 0.2.7. The category of groups and group homomorphisms, denoted $\mathbb{GRP}$.

•In this category the collection of object is taken to be the class of groups.
•The morphisms are taken to be group homomorphisms, where the domain and codomain of a group homomorphism is taken to be the domain and codomain of the homomorphism considered as a function to the underlying sets.
• The identity morphism, is the identity homomorphism which sends every object in a group G, to itself.
• Associativity follows from associativity of the underlying set functions.

◇

Example 0.2.8. The category of abelian groups and group homomorphisms, denoted $\mathbb{ABL}$.

• The collection of objects is taken to be the class of all abelian groups.
• The morphisms between abelian groups are however just group homomorphisms.
• Composition of group homomorphisms remains well defined from the above proposition.

• The identity homomorphism acts as the identity morphism.
• Finally, associativity holds due to associativity of the underlying set functions.

◇

Definition 0.2.9. *A topological space is a pair (X, V), where X is a set, and V is a topology, i.e. a collection of subsets of X, that are taken to be the open sets of the topological space, in order to qualify as a topology, the open sets have to satisfy the following axioms:*
• $\emptyset \in V$
• $X \in V$
• $A \cap B \in V$, *for all* A, $B \in V$.
• $\bigcup_{i=1}^{\infty} A_i \in V$, *for all* $A_i \in V$.

▷

Definition 0.2.10. *A continuous function f between two topological spaces (X, V), and (Y, W), is a function $f : X \to Y$, such that for all $w \in W$, $f^{-1}[w] \in V$. Where $f^{-1}[w] = \{x \in X \mid f(x) \in V\}$.*

▷

Proposition 0.2.11. *Given two continuous functions $f : X \to Y$, $g : Y \to Z$, mapping between topological spaces (X, U), (Y, V), (Z, W), the function $g \circ f$ is a continuous function, mapping between the topological spaces (X, U) and (Z, W).*

Proof. We need to show that for all $w \in W$:

$$(g \circ f)^{-1}[w] \in U.$$

But $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, hence:

$$(g \circ f)^{-1}[w] = f^{-1}(g^{-1}[w]).$$

We know that g is continuous, that is, $g^{-1}[w] = v$ for some $v \in V$, therefore:

$$f^{-1}(g^{-1}[w]) = f^{-1}[v].$$

But once again f is continuous, so $f^{-1}[v] = u$ for some $u \in U$, giving us the result that:

$$(g \circ f)^{-1}[w] \in U.$$

Hence $(g \circ f)$ is continuous. □

Example 0.2.12. The category of topological spaces and continuous maps, denoted $\mathbb{TOP}$.
• This category takes all topological spaces as its collection of objects.
• The morphisms are continuous maps between the topological spaces.
• The domain and codomain of a morphism are taken to be the domain and codomain of the morphism considered as a set function.
• The above proposition tells us that composition is well defined.
• Finally associativity follows from function associativity.

◇

Remark 0.2.13. We are now going to work through two further examples of categories that are of the form 'structured sets and morphisms', the difference is that with the following examples the structures may be unfamiliar, however for reasons that will become clear later on this this section, it would be a good idea to persevere and work through them anyway.

Definition 0.2.14. *A monoid is a triple* $(X, \cdot, e_X)$*, where* X *is a set,* $\cdot$ *is a binary operation,*

$$\cdot : X \times X \to X$$

Moreover e_X *is an identity element i.e. an element such that* $\forall x \in X$*:*

$$e_X \cdot x = x = x \cdot e_X.$$

Finally, the binary operation is associative, $\forall x, y, z \in X$*:*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

▷

Definition 0.2.15. *A monoid homomorphism between monoids,* $(X, \cdot, e_X)$ *and* $(Y, *, e_Y)$ *is a function* $f : X \to Y$ *such that,* $\forall a_1, a_2 \in X$*:*

$$f(a_1 \cdot a_2) = f(a_1) * f(a_2).$$
$$f(e_X) = e_Y.$$

▷

Proposition 0.2.16. *The composition of two monoid homomorphism is a monoid homomorphism.*

Proof. Given monoid homomorphisms, $f : M \to N$, $g : N \to P$, we need to check that:

$$(g \circ f)(e_M) = e_P.$$
$$(g \circ f)(m_1 \cdot m_2) = (g \circ f)(m_1) \cdot (g \circ f)(m_2).$$

To see the first requirement:

$$(g \circ f)(e_M) = g(f(e_M)) = g(e_N) = e_P.$$

And the second:

$$(g \circ f)(m_1 \cdot m_2) = g(f(m_1 \cdot m_2)) = g(f(m_1) \cdot f(m_2)) = gf(m_1) \cdot gf(m_2).$$

□

Example 0.2.17. The category of monoids and homomorphisms, denoted as $\mathbb{MON}$.

- $Obj\ C$ is the class of all monoids.
- $Mor\ C$ is the class of all monoid homomorphisms.
- For a given morphism, the domain and codomain of the morphism, are the domain and codomain of the homomorphism.
- From the above proposition, we see that composition is well defined.
- The identity homomorphism is the identity morphism.
- Finally, homomorphism composition is known to be associative.

We will come back to this example later on, as there is an interesting relation between monoids and categories.

◇

We now give a brief account of some of the orderings it is possible to give on a set, though for the moment only the concept of a partial ordering will be taken any further.

Definition 0.2.18. *A pre-order on a set X, is a relation $\leq$, that satisfies the following conditions:*
• *Reflexivity:* $x \leq x, \forall x \in X$
• *Transitivity:* $x \leq y \leq z \implies x \leq z, \forall x, y, z \in X$

▷

Definition 0.2.19. *A partial order on a set X, is a relation $\leq$, such that $\leq$ is a pre order that also satisfies:*
• *Antisymmetry:* $(x \leq y \text{ and } y \leq x) \implies x = y, \forall x, y \in X$

▷

Definition 0.2.20. *A total order on a set X, is a relation $\leq$, where $\leq$ is a pre-order, such that:*
• *Totality :* $x \leq y$ *or* $y \leq x$.

▷

Remark 0.2.21. A total order is a partial order, and a partial order is a pre order.

Example 0.2.22. Posets and Monotone Functions, $\mathbb{POS}$.

A poset is a partially ordered set, or in the above terminology, a pair $(X, \leq)$, where X is a set and $\leq$ is a partial order on X. A monotone function between posets $(X, \leq)$ and $(Y, \preceq)$ is a function $f : X \to Y$ such that:

$$(x_1 \leq x_2) \implies (f(x_1) \preceq f(x_2)) \ \forall x_1, x_2 \in X$$

It turns out that posets and monotone functions form a category, the only non-trivial thing to verify is that the composition of two monotone functions is itself a monotone function.
Suppose $f : X \to Y$ and $g : Y \to Z$ are monotone functions between $(X, \leq)$, $(Y, \preceq)$, $(Z, \trianglelefteq)$, then:

$$(x_1 \leq x_2) \implies f(x_1) \preceq f(x_2) \implies (g(f(x_1))) \trianglelefteq (g(f(x_2))) \implies (g \circ f(x_1)) \trianglelefteq (g \circ f(x_2)).$$

Therefore $g \circ f$ is a monotone function.

◇

We now provide a list of more categories in this vein, i.e. structured sets and maps that preserve the structure, we provide these examples without proof.

• Rings and Ring Homomorphisms.
• Fields and Field Homomorphisms.

• Differentiable Manifolds and Smooth maps.
• Right Modules and Module Homomorphisms.
• Vector Spaces and Linear Transformations.

Now that we have seen a few examples of categories, we might think that we have a pretty good intuitive grasp of what a category is, however there is a depth to the definition that is not immediately apparent, the next example explores this idea.

Example 0.2.23. $\mathbb{N}$ can be given the structure of a category.
• Let *Obj* $\mathbb{N}$ be a single object(what the object 'is' is unimportant, for this example just let it be Ø, the empty set).
• Mor $\mathbb{N}$ is just $\mathbb{N}$.
• The domain and codomain of each $n \in \mathbb{N}$ is Ø.
• The interesting idea is that: $n \circ m = n + m$.
• The identity morphism is 0, we see that $0 \circ n = 0 + n = n$, and $n \circ 0 = n + 0 = n$.
• The composition of two morphisms is clearly another morphism(as adding two natural number produces another natural number).
• Finally composition of morphisms is associative, this follows from the associativity of addition.

◇

Remark 0.2.24. There are two interesting things to note here, the first is that the object in this category is actually irrelevant, not only could it be replaced by any other object, but all the structure still exists without the object at all.

Secondly, the reason $\mathbb{N}$ can be described in the language of categories is that it is a monoid, and generally a monoid gives rise to a one element category and vice versa, we investigate this in the following proposition.

Proposition 0.2.25. *There is a correspondence between one element categories and monoids.*

Proof. For a monoid, $(X, \cdot, e)$ define a one element category C by:

• *Obj* C is X (though we could use any other object).
• Mor C is taken to be the set X.
• The domain and codomain of any element of X is taken to be X.
• Composition of morphisms is given by the product within the monoid, i.e. $x_1 \circ x_2 = x_1 \cdot x_2$, this composition necessarily exists due to the fact that $\cdot$ is a binary operation and hence the set X is closed under it.
• The identity element is seen to function as the identity morphism.

$$e \circ x = ex = x = xe = x \circ e$$

• Finally, the associativity of composition of morphisms from associativity of the binary operation within the monoid.

For a one element category, C, we define a monoid $(MorC, \circ, e)$ where e is the identity morphism on the element of C, this is clearly a monoid, we can see this by running through the above proposition and seeing that all the arguments work backwards as well. $\square$

Proposition 0.2.26. *There is a correspondence between pre-ordered sets and categories with at most one morphism between two objects.*

Proof. Given a pre order $(P, \leq)$, we define a category C which has $Obj\ C = P$, and then has a morphism $f : a \to b$ iff $a \leq b$. It may not be immediately clear that this is in fact a category, but note that reflexivity implies the existence of an identity morphism, transitivity of $\leq$ ensures that morphism composition is well defined. Also, the fact that for any two objects there is at most one morphism between them, implies that associativity is satisfied.

Given a category that has the property that between any two objects there is at most one morphism, we define a pre-order by reversing the above construction, i.e. use the *obj* C as the underlying set, and construct a pre-order, by saying $a \leq b$ iff there exists a morphism $f : a \to b$. As above reflexivity is implied by the existence of an identity morphism, and transitivity comes from morphism composition. Finally, antisymmetry follows from the fact that between any two objects there is at most one morphism, hence the only times that we can have morphisms $f : A \to B$ and $g : B \to A$ are when $A = B$ and $f = g = id_A$.
$\square$

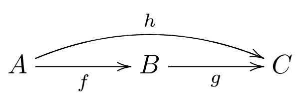
Proposition 0.2.27. *For any object A of a category $\mathcal{C}$, the identity morphism id_A is uniquely determined.*

Proof. This proof is much the same as the proof that the identity element in a group is unique. Let $id_A : A \to A$ and $id_A{}' : A \to A$ be identity morphisms on A, then $id_A = id_A \circ id_A{}' = id_A{}'$. $\square$

A useful aid to thinking 'categorically' is to construct diagrams. A diagram is a directed graph that is labeled consistently with a section of a category, the verticies corresponding to objects, and morphisms corresponding to arrows between the domain and codomain of the morphism. It is a strongly established convention that all diagrams should commute.

Remark 0.2.28. A directed graph has a technical definition as a collection of vertices, and a collection of directed edges (ordered pairs of vertices), but this isn't really too important, once you have seen a couple of diagrams, you quickly understand what they are and how they work.

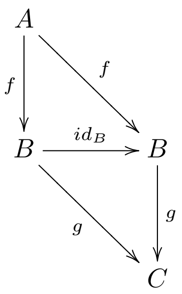
So to say that the following diagram commutes:



Would be the statement that:

$$g \circ f = h.$$

We can therefore encode the identity requirement for a category by saying that for all $f : A \to B$, and $g : B \to C$ the following diagram commutes:

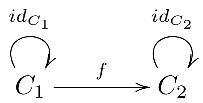


We now present a few examples of finite categories, making use of diagrammatic reasoning.

Example 0.2.29. The category **1** consisting of the single object C, and one morphism, we know that in order that $\mathcal{C}$ be a category we need to ensure that C has an identity morphism, hence the single morphism must be the identity morphism on C, id_C, therefore $\mathcal{C}$ can be represented as:

◇

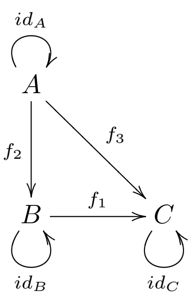
Example 0.2.30. The category **2** comprises two objects and three morphisms.



Note that composition of morphisms is determined a priori.

◇

Example 0.2.31. The category **3**, has three objects, and six arrows:



Once again, composition of arrows is uniquely determined, $f_1 \circ f_2 = f_3$, and identities work in the obvious way.

◇

Definition 0.2.32. *A discrete category is a category where every morphism is an identity morphism. Note that a discrete category is also a pre-order, due to the uniqueness of the identity morphism. In fact it is the pre-order on a set where ($a \leq b$ iff $a = b$).*

▷

Example 0.2.33. The category **1** is a discrete category.

◇

Definition 0.2.34. *A category $\mathcal{C}$ is said to be a small category if Obj $\mathcal{C}$ and Mor $\mathcal{C}$ are both sets. Otherwise the category is said to be large.*

▷

Remark 0.2.35. If you have never studied set theory then the only thing you need to keep in mind is that not all collections are sets, if a collection is too big then it is a class. Most things used in practice are sets, the collection of all functions $\mathbb{N} \to \mathbb{R}$ is a set for example, as is $\mathbb{C}$, the only things you are likely to come up against that are not sets are things like the collection of all sets, or the collection of all groups.

Examples 0.2.36. $\mathbb{SET}$, $\mathbb{MON}$, $\mathbb{GRP}$ are all large. The categories **1**, **2**, and **3** are all finite and therefore small.

Remark 0.2.37. It might appear that all the interesting categories are large, for this reason we make use of an intermediate condition, one which will still allow us to carry out fruitful inquiry, but will also include more of the categories that we wish to study.

Definition 0.2.38. *A category $\mathcal{C}$ is locally small if for all pairs of objects A, B in $\mathcal{C}$, the collection:*

$$Hom_{\mathcal{C}}(A, B) = \{f \in Mor\mathcal{C} \mid f : A \to B\}$$

is a set. These collections are called hom sets.

▷

Definition 0.2.39. *By $Hom_{\mathcal{C}}(A, B)$ we denote the collection of all morphisms in the category $\mathcal{C}$ from the object A to the object B. The term 'Hom Set' is generally used to denote this collection, this is because in any locally small category, this collection will indeed be a set. If there is no chance of confusion we will sometimes write $\mathcal{C}(A, B)$.*

▷

Remark 0.2.40. Most useful categories that we want to study are locally small. Also note that all small categories are locally small.

Example 0.2.41. $\mathbb{SET}$ is locally small, this is because the collection of functions between two sets is itself a set.

◇

Definition 0.2.42. *We say that $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$ when:*
- *$\mathcal{D}$ is a category.*
- *Obj $\mathcal{D} \subseteq$ Obj $\mathcal{C}$*
- *$Hom_{\mathcal{D}}(A, B) \subseteq Hom_{\mathcal{C}}(A, B)$*
- *The identity morphisms and composition of morphisms are the same in both categories.*

▷

Example 0.2.43. The category of abelian groups is a subcategory of the category of groups. Too see this we simply need to note that every abelian group is also a group.
• Hence $Obj\ \mathbb{ABL} \subseteq Obj\ \mathbb{GRP}$.
• Furthermore, the group homomorphism between two abelian groups are exactly the same depending on whether the abelian group is considered as an abelian group or just a group. Therefore, $Hom_{\mathbb{ABL}}(A, B) = Hom_{\mathbb{GRP}}(A, B)$, whence $Hom_{\mathbb{ABL}}(A, B) \subseteq Hom_{\mathbb{GRP}}(A, B)$.
• Finally as the homomorphisms are equal in either case, we can conclude that composition and identities are equal.

◇

0.2.1 Constructing New Categories.

In this section we examine some of the constructions that allow us to build new categories from old categories. The next definition looks a lot like the cartesian product of two sets, and there is a deep relation between the two, however the actual mechanics of forming the product category may be very different from what goes on in forming the product of two sets.

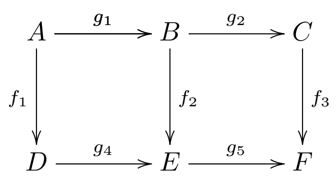
Definition 0.2.44. *Given two categories, $\mathcal{C}$ and D, we can form the product category $\mathcal{C} \times \mathcal{D}$, the objects of $\mathcal{C} \times D$ are pairs (A, B) such that A is an object in $\mathcal{C}$ and B is a object in $\mathcal{D}$. Morphisms are taken to be pairs of morphisms (f, g) such that f is a morphism in $\mathcal{C}$ and g is a morphism in $\mathcal{D}$.*

Morphism composition in $C \times D$ is done component wise, with respect to composition in $\mathcal{C}$ and $\mathcal{D}$. That is for $(f, g) : A \times B \to C \times D$ and $(h, i) : C \times D \to E \times F$, we take $(h, i) \circ (f, g) : A \times B \to E \times F$ to be:

$$(h \circ f, i \circ g).$$

▷

Lemma 0.2.45. *If two the 'inside squares' commute, then the outside rectangle commutes.*



Proof. We have that

$$g_4 \circ f_1 = f_2 \circ g_1$$
$$g_5 \circ f_2 = f_3 \circ g_2.$$

We want that:

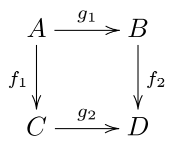
$$g_5 \circ g_4 \circ f_1 = f_3 \circ g_2 \circ g_1$$

Therefore:

$$(g_5 \circ g_4) \circ f_1 = g_5 \circ (g_4 \circ f_1) = g_5 \circ (f_2 \circ g_1) = (g_5 \circ f_2) \circ g_1 = (f_3 \circ g_2) \circ g_1 = f_3 \circ (g_2 \circ g_1)$$

□

Definition 0.2.46. *The arrow category $\mathcal{C}^{\rightarrow}$, is formed by taking the morphisms of the category $\mathcal{C}$ as objects, and then specifying a morphism (in $\mathcal{C}^{\rightarrow}$) between two objects $f_1 : A \rightarrow B$ and $f_2 : A \rightarrow B$ to be an ordered pair of $\mathcal{C}$ morphisms (g_1, g_2) such that:*

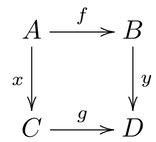


▷

Proposition 0.2.47. *The arrow category is a category.*

Proof. • We have a well defined collections of objects and morphisms.
• Morphism composition is done component wise, and the preceding lemma ensures that this is well defined.
• The identity morphism on $f : A \rightarrow B$ is seen to be (id_A, id_B), which clearly exists for any morphism.
• Associativity in $\mathcal{C}^{\rightarrow}$ follows simply from associativity in $\mathcal{C}$. □

Example 0.2.48. The category $\mathbb{SET}^{\rightarrow}$ is formed of functions $f : A \rightarrow B$, then a morphism between functions $f : A \rightarrow B$, and $g : C \rightarrow D$, is a pair of functions (x, y), where $x : A \rightarrow C$, and $y : B \rightarrow D$, such that $y \circ f = g \circ x$.



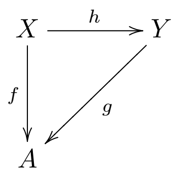
◇

Recall that every group is also a monoid, therefore we can think of a group as category by using the construction outlined above.

Example 0.2.49. If we take a group G, and then consider it as a category, then $G^{\rightarrow}$, is the category comprised of elements of G, and a morphism between two elements g_1 and g_2, is a pair of elements (x, y), such that $yg_1 = g_2x$.

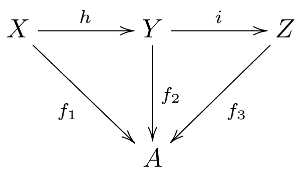
◇

Definition 0.2.50. *Given a category $\mathcal{C}$, and an object A of $\mathcal{C}$, we can form the slice category $(\mathcal{C}, A)$, it is the category of all morphisms in $\mathcal{C}$, with codomain A. Then a morphism between these objects (morphisms, $f : X \rightarrow A$, and $g : Y \rightarrow A$), is an arrow $h : X \rightarrow Y$, such that $g \circ h = f$, this encapsulated in the following diagram:*



▷

Lemma 0.2.51. *In a category $\mathcal{C}$, if we have the following morphisms and objects:*



If we have that:

$$f_2 \circ h = f_1.$$
$$f_3 \circ i = f_2.$$

Then:

$$f_3 \circ (i \circ h) = f_1.$$

Proof. $f_3 \circ (i \circ h) = (f_3 \circ i) \circ h = f_2 \circ h = f_1$. □

Proposition 0.2.52. *The slice category is a category.*

Proof. • We have a well defined collection of objects.
• We also have a well defined collection of morphisms with clear domains and codomains.
• Take composition of morphisms to be composition in the category $\mathcal{C}$, the above lemma tells us that this is well defined.
• For a morphism $f : X \to A$, the morphism id_X, acts as an identity morphism in $(\mathcal{C}, A)$.
• Associativity follows from associativity in $\mathcal{C}$. □

Example 0.2.53. If we start with the category $\mathbb{SET}$, then we can take the object $\mathbb{R}$, and form the slice category $\mathbb{SET}/\mathbb{R}$. This category is comprised of functions which map into $\mathbb{R}$, then a morphism between functions $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ is a function $\alpha : X \to Y$ such that $g \circ \alpha = f$.

◇

Definition 0.2.54. *If we start with a set X, then we can form the discrete category $D(X)$, this comprises:*
• *The collection of objects is simply taken to be the set X.*
• *For each element of the set we define an identity morphism, these are taken to be all the morphisms.*
• *The only morphism composition that can occur is when we compose an identity morphism with itself, this is always defined, and moreover is predetermined.*
• *The existence of identity elements is given above.*
• *Finally associativity of composition follows trivially from the fact that we only have identity morphisms.*

▷

0.2.2 Morphisms.

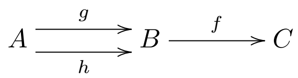
In $\mathbb{SET}$, not all morphisms are equal, some have special properties, two such properties are injectivity and surjectivity. Remember that for a function $f : X \to Y$,

$$\text{Injective} \iff f(x_1) = f(x_2) \implies x_1 = x_2.$$
$$\text{Surjective} \iff \forall y \in Y,\ \exists x \in X, \text{ such that } f(x) = y.$$

Definition 0.2.55. *In a category $\mathcal{C}$, a morphism $f : B \to C$ is a monomorphism if for all pairs of morphisms, $g : A \to B$, $h : A \to B$,*

$$(f \circ g = f \circ h) \implies g = h.$$

Whenever we have two morphisms that map between the same two objects we call them parallel morphisms.



So in the above diagram, g and h are parallel morphisms.

▷

Proposition 0.2.56. *In $\mathbb{SET}$, a function is injective iff it is a monomorphism.*

Proof. First assume that $f : Y \to Z$ is injective, and let $g : X \to Y$ $h : X \to Y$, be two functions, such that:

$$f \circ g = f \circ h.$$

More explicitly this means:

$$(f \circ g)(x) = (f \circ h)(x),\ \forall x \in Y.$$

Hence from the definition of composition:

$$f(g(x)) = f(h(x)),\ \forall x \in Y.$$

But from f being injective, this implies:

$$g(x) = h(x),\ \forall x \in Y$$

Therefore $g = h$, and f is a monomorphism.

Now suppose that f is a monomorphism, and assume for contradiction that f is not injective, then for some $x_1, x_2 \in X$:

$$f(x_1) = f(x_2), \text{ but } x_1 \neq x_2.$$

Then define functions: $g : \{0\} \to Y$ and $h : \{0\} \to Y$ where $g(0) = x_1$ and $h(0) = x_2$, Hence:

$$(f(x_1) = f(x_2)) \implies f(g(0)) = f(h(0)) \implies (f \circ g = f \circ h), \text{ but } g \neq h$$

Therefore f is not a monomorphism, and we have the required result. □

Definition 0.2.57. *A morphism $f : X \to Y$ is an epimorphism if for all pairs of parallel morphisms, $g : Y \to Z$, $h : Y \to Z$,*

$$(g \circ f = h \circ f) \implies g = h.$$

▷

Example 0.2.58. In $\mathbb{N}$ (considered as a category), all arrows are monomorphisms and epimorphisms. To see this, simply note that $\forall a, b, c \in \mathbb{N}$:

$$a + c = b + c \implies a = b$$
$$c + a = c + b \implies a = b,$$

◇

Proposition 0.2.59. *In* $\mathbb{SET}$ *a function is an epimorphism iff it is surjective.*

Proof. Suppose that $f : X \to Y$ is a not surjection, then $\exists y_1 \in Y$ such that $\nexists x \in X$ with $f(x) = y_1$, now define two functions

$$g : Y \to \{z_1, z_2\},\ h : Y \to \{z_1, z_2\}$$

where $z_1 \neq z_2$, by $g(y_1) = z_1$ and $h(y_1) = z_2$, but $g = h$ elsewhere, then:

$$h \circ f = g \circ f, \text{ but } h \neq g.$$

Therefore f is not an epimorphism.

Now suppose that f is a surjective and that we have he following set up:

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

Assume that $gf = hf$, but $g \neq h$, this means that $\exists\ b_0 \in B$, such that $g(b_0) \neq h(b_0)$. We know that f is surjective, therefore, $\exists a_0 \in A$ such that $f(a_0) = b_0$. but then we know that $gf(a_0) = hf(a_0)$, and therefore, $g(b_0) = h(b_0)$ and therefore by contradiction, we must have that $g = h$, and hence f is an epimorphism.

□

Example 0.2.60. An example of a morphism that is not an epimorphism is the set function: $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 0$, for all $x \in \mathbb{R}$. This follows straight away from the above proposition.

◇

Proposition 0.2.61. *All morphisms in a poset are monomorphisms and epimorphisms.*

Proof. We know that in a poset there is at most one morphism between any two given objects, hence for any two parallel morphisms,

$$g : X \to Y,\ h : X \to Y.$$

We can say that they are equal. Therefore the implication:

$$fg = fh \implies g = h$$

Is never false. Likewise for the implication:

$$gf = hf \implies g = h$$

Therefore all morphisms are both epimorphisms and monomorphisms. □

Definition 0.2.62. *A morphism $f : X \to Y$ is an isomorphism if there exists an inverse morphism, that is a morphism $g : Y \to X$ such that:*

$$f \circ g = e_X \text{ and } g \circ f = e_Y.$$

▷

Example 0.2.63. In $\mathbb{TOP}$, a continuous function $f : V \to W$ is an isomorphism iff there exists another continuous function $g : W \to V$ such that $f \circ g = id_V$, and $g \circ f = id_W$. However this is just the definition of a homeomorphism, therefore in the $\mathbb{TOP}$, a continuous function is is an isomorphism iff it is a a homeomorphism.

◇

Proposition 0.2.64. *If a morphism is invertible, then the inverse is unique.*

Proof. Suppose that $f : X \to Y$ is invertible, and let g_1 and g_2 be two morphisms such that:

$$f \circ g_1 = f \circ g_2 = e_X \text{ and } g_1 \circ f = g_2 \circ f = e_Y.$$

Then:

$$g_1 = e_Y \circ g_1 = (g_2 \circ f) \circ g_1 = g_2 \circ (f \circ g_1) = g_2 \circ e_X = g_2.$$

Hence we are warranted in denoting the inverse of f by f^{-1}. □

Proposition 0.2.65. *Any isomorphism is a monomorphism and an epimorphism*

Proof. We first show that it is an epimorphism. Let $f : X \to Y$ be an isomorphism with inverse $g : Y \to X$, and let $h_1 : Y \to Z$, $h_2 : Y \to Z$ be morphisms where:

$$h_1 \circ f = h_2 \circ f.$$

Then multiplying on both sides by g, gives us:

$$(h_1 \circ f) \circ g = (h_2 \circ f) \circ g$$

Rearranging brackets tells us that:

$$h_1 \circ (f \circ g) = h_2 \circ (f \circ g)$$

But as f and g are inverses, we know that:

$$f \circ g = e_X.$$

Hence, we get that:

$$h_1 \circ e_X = h_2 \circ e_X \implies h_1 = h_2$$

And therefore, f is an epimorphism.

The proof that f is a monomorphism works in a very similar way. If we let $h_1 : Y \to X$, $h_2 : Y \to X$ be morphisms where:

$$f \circ h_1 = f \circ h_2.$$

Then as above, multiply both sides by g, giving:

$$g \circ (f \circ h_1) = g \circ (f \circ h_2).$$

Now from associativity and the fact that $g \circ f = e_Y$, we have:

$$e_Y \circ h_1 = e_Y \circ h_2.$$

Hence, $h_1 = h_2$, and f is an monomorphism. □

In the case of $\mathbb{SET}$, a morphism is an isomorphism (bijection) exactly when it is a monomorphism (injection) and an epimorphism (surjection), however this fact does not hold in a general category.

Proposition 0.2.66. *There exists a morphism f in $\mathbb{N}$ such that f is an epimorphism and a monomorphism, but is not an isomorphism.*

Proof. Note that in $\mathbb{N}$, all morphisms are left and right cancellable, to see this, simply note that $n + m = k + m \implies n = k$, and $n + m = n + l \implies m = l$. This means that all morphisms are both monomorphisms and epimorphisms. However the only natural number with an additive inverse in $\mathbb{N}$ is 0, therefore 0 is the only isomorphism. Hence, for example, 1 is a monomorphism and an epimorphism but not an isomorphism. □

It is said that in Category Theory, objects are only classified up to isomorphism, this corresponds to the fact that within a given category, the goal is often to classify objects up to isomorphism, in topology the goal is to classify all topological spaces up to isomorphism, in group theory, it is to classify all groups up to isomorphism, and so on. To that end, we give a characterization of isomorphism within an abstract category.

Definition 0.2.67. *In a category $\mathcal{C}$, we say that two objects; A, B, are isomorphic if there exists an isomorphism $f : A \to B$*

▷

Example 0.2.68. In $\mathbb{SET}$, two objects are isomorphic to each other if there exists an isomorphism between them, but as we saw above, in $\mathbb{SET}$ this corresponds simply to them being bijective, hence, in $\mathbb{SET}$ any two bijective sets are isomorphic.

◇

0.2.3 Duality

Let us recall the original definition of a category:

A category $\mathcal{C}$ comprises:

- A collection of objects, denoted, $Obj\ \mathcal{C}$.
- A collection of morphisms, denoted, $Mor\ \mathcal{C}$.
- Two operations, domain and codomain, that for each $f \in Mor\ C$, assign objects; $dom(f)$, $cod(f)$, called the domain and codomain of f, if we have $A = dom(f)$, $B = cod(f)$, then we write, $f : A \to B$.
- For each pair of morphisms, f, g, such that, $f : A \to B$ and $g : B \to C$, there exists a morphism $g \circ f : A \to C$.

• For each $A \in Obj\ \mathcal{C}$ there exists a morphism $id_A : A \to A$, such that for all $f : A \to B$ and $g : C \to A$, $f \circ id_A = f$ and $id_A \circ g = g$.
• For all $f : A \to B$, $g : B \to C$, $h : C \to D$; $(h \circ g) \circ f = h \circ (g \circ f)$. This is the associativity requirement.

If we have a category $\mathcal{C}$, then we can form a new category which we will denote $\mathcal{C}^{op}$, called the opposite category, we build it by taking all the objects and morphism of the original category:
• A collection of objects, in this case: $Obj\ \mathcal{C}$.
• A collection of morphisms, in this case: $Mor\ \mathcal{C}$.
But we now turn all the morphisms in the opposite direction, that is if for a morphism f in $\mathcal{C}$, $dom(f) = A$ and $cod(f) = B$, then in $\mathcal{C}^{op}$, we set $dom(f^{op}) = B$ and $cod(f^{op}) = A$.

Note that from the point of view of the collections $Mor\mathcal{C}$, and $Mor\mathcal{C}^{op}$, f and f^{op} are the same object, however from the point of view of the category they are different because they have different domains and codomains, hence we add on the ^{op} to the f to make it clear from which category we are viewing it.

Diagrammatically we are making the following change:

$$A \xrightarrow{f} B \qquad Becomes: \qquad A \xleftarrow{f^{op}} B$$

We can see that this gives us a clearly defined collection of objects, a clearly defined collection of morphisms, and a domain and codomain for each morphism, we still need to check the remaining axioms, namely, composition, identity and associativity.

If we take two morphisms in $\mathcal{C}^{op}$, $f^{op} : A \to B$ and $g^{op} : B \to C$, then these correspond to two morphisms in $\mathcal{C}$, $f : B \to A$, $g : C \to B$, we wish to take the morphisms f^{op} and g^{op} and form a new morphism $g^{op} \circ f^{op} : A \to C$ in $\mathcal{C}$, therefore we define $g^{op} \circ f^{op}$ to be morphism $(f \circ g)^{op}$. Clearly as $f \circ g : C \to A$, $(g \circ f)^{op} : A \to C$ and hence has the correct domain and codomain. Moreover, we know that this morphism always exists due to the fact that the composition in the original category, $\mathcal{C}$, always exists.

We now need an identity morphism for every object. There only really seems to be one choice that we can take, and that is for a given object A, simply taking the identity morphism id_A, because id_A has the same domain and codomain, we could be sloppy and say that the identity morphism for an object A in $\mathcal{C}^{op}$ is just the identity morphism of A in $\mathcal{C}$, however to be perfectly rigorous, we should say instead that $(id_A)^{op}$ is the identity morphism. Of course we haven't actually shown that it has the required properties, to show this we need to show that for all morphisms $f^{op} : A \to B$ and $g^{op} : B \to C$

$$(id_B)^{op} \circ g^{op} = g^{op} \text{ and } f^{op} \circ (id_B)^{op} = f^{op}.$$

To see this we put in the definition of composition in this category, hence:

$$(id_B)^{op} \circ g^{op} = (g \circ id_B)^{op} = (g)^{op}.$$
$$f^{op} \circ (id_B)^{op} = (id_B \circ f)^{op} = f^{op}.$$

So we see that $(id_B)^{op}$ is the identity morphism for B in $\mathcal{C}^{op}$.

Finally we need to show that composition of morphisms is associative. That is, for all morphisms $f^{op} : A \to B$, $g^{op} : B \to C$, $h^{op} : C \to D$:

$$(h^{op} g^{op}) f^{op} = h^{op}(g^{op} f^{op})$$

To see this:

$$(h^{op} \circ g^{op}) \circ f^{op} = (g \circ h)^{op} \circ f^{op} = (f \circ (g \circ h))^{op} = ((f \circ g) \circ h)^{op} = h^{op} \circ (f \circ g)^{op} = h^{op} \circ (g^{op} \circ f^{op}).$$

Therefore we see that from $\mathcal{C}$ we have formed a new category $\mathcal{C}^{op}$. We give the formal definition of this category below.

Definition 0.2.69. *The opposite/dual category C^{op}, of the category $\mathcal{C}$, is the category formed by retaining the classes Obj $\mathcal{C}$ and Mor $\mathcal{C}$, but reversing the direction of all morphisms, so that if $f : A \to B$ in $\mathcal{C}$ then $f^{op} : B \to A$ in C^{op}. The composition of two morphisms $f^{op} : B \to A$ and $g^{op} : C \to B$ in C^{op} is defined exactly if $g \circ f$ is defined, in which case we say that $f^{op} \circ g^{op} = (g \circ f)^{op}$. Note that domain$(f^{op})$ = codomain(f) and domain(f) = codomain(f^{op}).*

▷

Example 0.2.70. The category $\mathbb{SET}^{op}$ has as objects all sets, and morphisms all set functions, but with domain and codomain reversed.

◇

Example 0.2.71. If $\mathcal{C}$ is a discrete category, then $C^{op} = C$.

◇

Remark 0.2.72. For any category $\mathcal{C}$, $(C^{op})^{op} = C$.

You might ask, why we have gone to such great lengths to define this category? The following definition justifies the trouble we have gone to.

Definition 0.2.73. *If Σ is a statement in a category $\mathcal{C}$, then the dual of Σ, Σ^{op}. Is the statement obtained by replacing each instance of 'dom' by 'cod', each instance of 'cod' by 'dom', and '$h = f \circ g$' by '$h = g \circ f$'*

▷

Remark 0.2.74. The dual of a statement can be thought of as the original statement but applied to the dual category. Therefore whenever we prove a statement in category theory, we also prove the dual of the statement.

Example 0.2.75. The epimorphisms and monomorphisms are dual.

◇

Remark 0.2.76. If you can't quite grasp what is going on in the above definition then it's best to come back later on. The following section contains lots of examples of duality, the idea itself is easy to grasp but hard to formalize.

0.2.4 Elementary Structures Within Categories.

Definition 0.2.77. *An initial object ϕ in a category $\mathcal{C}$, is an object such that for all objects A in $\mathcal{C}$ there exists a unique morphism $\phi_A : \phi \to A$, from the initial object, to the object A.*

▷

Proposition 0.2.78. *Initial objects are unique up to isomorphism.*

Proof. Let A and B be two initial objects $\mathcal{C}$, from the initial property of A,

$$\exists!\ f : A \to B,$$

And from the initial object property of B:

$$\exists!\ g : B \to A,$$

Now if we compose these two morphisms we get:

$$g \circ f : A \to A, \text{ and } f \circ g : B \to B.$$

The trick now is to note that we can apply the initial object property of A to A, to conclude that:

$$\exists!\ h : A \to A.$$

We now have a total of three morphisms $A \to A$, namely:

$$id_A : A \to A,\ g \circ f : A \to A \text{ and } h : A \to A.$$

However we also know that h is the unique morphism mapping $A \to A$, allowing us to conclude that $id_A = g \circ f = h$.

We can now apply the same argument to $f \circ g$, i.e. that as $f \circ g : B \to B$, and $id_B B \to B$, where B is an initial object, $f \circ g = id_B$. And hence, f is an isomorphisms, and $A \cong B$. □

Example 0.2.79. In $\mathbb{SET}$, $\emptyset$ is the unique initial object.

To see this, note that for any $\mathbb{SET}$ X, the empty function $f : \emptyset \to X$ is uniquely determined, and always exists. From the above proposition any initial object in $\mathbb{SET}$ must be isomorphic (bijective) with the empty $\mathbb{SET}$, and hence as there is only one $\mathbb{SET}$ with zero elements, $\emptyset$ is the unique initial object in $\mathbb{SET}$.

◇

Definition 0.2.80. *A least element in a poset* $(P, \leq)$ *is an element* $p_1 \in P$ *such that* $p_1 \leq p$, $\forall p \in P$

▷

Example 0.2.81. In a poset, $(P, \leq)$, an initial object is a least element. The requirement that there be at most one morphism from the initial object to every other object is vacuously true in a poset where between any two objects there is at most one morphism. Moreover an element p_1 is a least element iff:

$$(p_1 \leq p,\ \forall p \in P) \text{ iff } (\exists \text{ a morphism } f : p_1 \to p,\ \forall r \in P) \text{ iff } (p_1 \text{ is a least element in } P).$$

Note that the above proposition all tells us that least elements are unique up to isomorphism, yet in a poset, due to anti-symmetry, isomorphic object are equal. Therefore we can deduce that a least element in a poset (if it exists) is unique.

◇

The above two examples give us our first sense of the power of category theory, the proposition that initial objects are unique up to isomorphism has been used to show that two disparate objects (the emptyset and the least element in a poset) are unique up to isomorphism.

Definition 0.2.82. *A terminal object in a category is an object, denoted* 1*, such that for every object, there exists a unique morphism from the object to* 1*.*

▷

Remark 0.2.83. Initial and terminal objects are dual, this is because an object A is terminal in $\mathcal{C}$ iff A^{op} is initial in C^{op}.

Proposition 0.2.84. *Terminal objects are unique up to isomorphism.*

Proof. This follows simply from the above proposition after noting that terminal and initial object are dual. □

Example 0.2.85. In $\mathbb{SET}$ all singleton sets, $\{a\}$ are terminal objects. This is because for any set X, there exists a unique function, $f : X \to \{a\}$ defined by $f(x) = a$, $\forall x \in X$.
We know that all terminal objects are isomorphic, hence we would expect that if a terminal object in $\mathbb{SET}$ exists it would be some class of bijective sets, moreover, once we have established that all singleton sets are terminal, uniqueness ensures that there are no other terminal objects in $\mathbb{SET}$, as any other set would have to be bijective with a singleton set, and hence itself a singleton.

◇

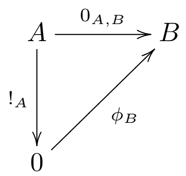
Definition 0.2.86. *A zero object is an object that is both initial and terminal.*

▷

Zero objects are of interest due to the fact that using the initial object, we can form a morphism between any two objects in the category.

Proposition 0.2.87. *If a category has a zero object, then there exists a 'zero morphism' between any two objects in the category. The zero morphism between* A *and* B *will be denoted* $0_{A,B}$*.*

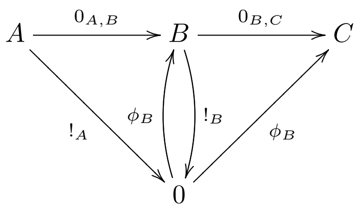
Proof. Suppose that we want to form a morphism between objects A and B, then using the initial property if 0, there exists a morphism $!_A : A \to 0$, and using the terminal property of 0, there exists a morphism $\phi_B : 0 \to B$, if we then take the composition of these morphisms, then $\phi_B \circ\, !_A : A \to B$ is a morphism from A to B.



□

Proposition 0.2.88. *The composition of two zero morphisms is a zero morphism.*

Proof. Assume we have he following set up:



Then the statement that the composition of zero morphisms is once again a zero morphism is the statement that:

$$\phi_B \circ\, !_B \circ \phi_B \circ !_A = \phi_B \circ !_A.$$

We know that 0 is a terminal object, hence $!_A : A \to 0$ is the unique morphism from A to 0, therefore as $(!_B \circ \phi_B \circ !_A) : A \to 0$, we have that,

$$!_A = 0_B \circ \phi_B \circ !_A.$$

Hence :

$$\phi_B \circ (0_B \circ \phi_B \circ !_A) = \phi_B \circ !_A.$$

□

Proposition 0.2.89. *In $\mathbb{GRP}$ the trivial group $\{e\}$ is a zero object. And the zero morphisms are the trivial homomorphisms. For two groups, G and H, the trivial homomorphism, is the homomorphism, which sends every element of G to the identity element of H.*

Proof. We first verify that $\{0\}$ is a terminal object. Let G be a group, then as G and $\{0\}$ are also $\mathbb{SET}$s, there exists a unique function $f : G \to \{0\}$, moreover this is a homomorphism, as $f(g_1 g_2) = 0 = 00 = f(g_1)f(g_2)$ and $f(e_G) = 0$.

We now need to show that $\{0\}$ is an initial object, to see this, let H be a group, then clearly the mapping $g : \{0\} \to H$ given by $g(0) = e_H$ is a homomorphism, moreover in order that it be a homomorphism it is required that it to send identities to identities, hence, any homomorphism must conform to this. This means that there is a unique morphism from $\{0\}$ to any group H.

Now that we have confirmed that $\{0\}$ is a zero object, we can see that this follows by simply composing the above two homomorphisms. □

Remark 0.2.90. We now give a generalisation of the process of constructing the cartesian product of two sets.

Definition 0.2.91. *Given two sets A, B the cartesian product $A \times B$ is the set of all ordered pairs where the first element is in A, and the second in B:*

$$A \times B = \{(a, b)|\ a \in A, b \in B\}.$$

▷

Remark 0.2.92. It might seem that it would be impossible to replicate this construction in a categorical setting where we only have access to functions and not 'elements'. However, note that to the set $A \times B$ there are associated two 'special functions' namely left and right projection:

$$p_1 : A \times B \to A,\ p_2 : A \times B \to B.$$

Defined by:

$$p_1((a,b)) = a \text{ and } p_2((a,b)) = b.$$

They are special in the sense that if we have some set C with a pair of maps,

$$f : C \to A,\ g : C \to B.$$

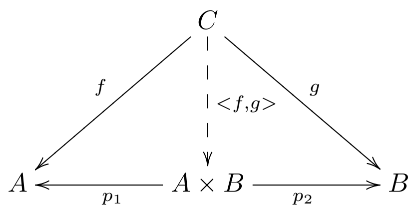
Then we define $< f, g >: C \to A \times B$ by:

$$< f, g > (x) = (f(x), g(x)).$$

We note that it have the properties:

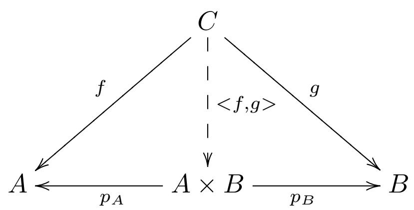
$$p_1 \circ < f, g >= f \text{ and } p_2 \circ < f, g >= g.$$

Moreover $< f, g >$ is the unique function that satisfies these conditions. This is encapsulated in the following diagram:



(there is a convention that a dotted line in a diagram denotes the unique such morphism that makes the diagram commute.)

Definition 0.2.93. *A product of two objects A and B in a category $\mathcal{C}$ is a $\mathcal{C}$ object $A \times B$ and two $\mathcal{C}$ morphisms ($p_A : A \times B \to A$, $p_B : A \times B \to B$) such that for any pair of morphisms of the form ($f : C \to A$, $g : C \to B$) there is a unique morphism (called the product morphism of f and g) $< f, g >: C \to A \times B$ such that the following diagram commutes:*



$\triangleright$

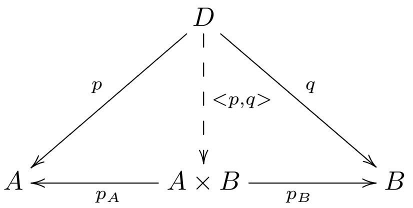
Remark 0.2.94. We said 'a' product rather than 'the' product because we have not yet shown that a product is unique. Note that in the case of $\mathbb{SET}$ both $A \times B$ and $B \times A$ are products of A and B, so we shouldn't expect complete uniqueness, we will instead show that a product is unique up to isomorphism.

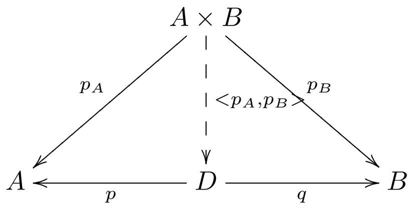
Proposition 0.2.95. *In a category $\mathcal{C}$, given two objects A and B, all products of A and B are isomorphic.*

Proof. Suppose we are given the objects A and B in a category $\mathcal{C}$ and are told that $A \times B$ is a product, now suppose that the following is also a product of A and B:

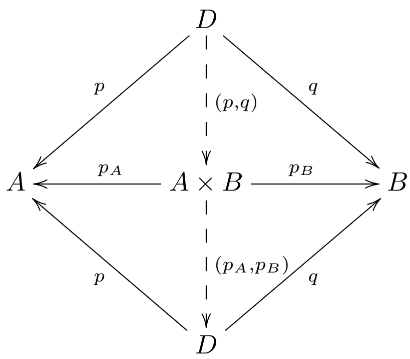
$$(D, p : D \to A, q : D \to B)$$

Then we can form two commutative diagrams:

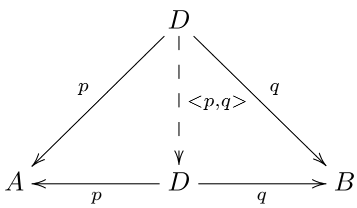




The first comes from considering $A \times B$ as a product, and using the product property on the two morphisms $p : D \to A$, $q : D \to B$ to get a morphism $< p, q >: D \to A \times B$. The second diagram comes from considering D as a product, and using the product property on the two morphisms $p_A : A \times B \to A$, $p_B : A \times B \to B$, to get a morphism $< p_A, p_B >: A \times B \to D$. We can combine the two into the following diagram:



We can see from this that $(p_A, p_B) \circ (p, q) : D \to D$. From the fact that D is a product, we can use the product property of D on itself. This tells us that there is a unique morphism $< p, q >: D \to D$ such that $q\circ < p, q >= q$ and $p\circ < p, q >= p$, this is encapsulated in the following diagram:



However, we can see that id_D fulfills this role, that is, $id_D : D \to D$, $p \circ id_D = p$ and $q \circ id_D = q$, hence it must be the case that $< p, q >= id_D$.
Recall that:

$$(p_A, p_B) \circ (p, q) : D \to D.$$

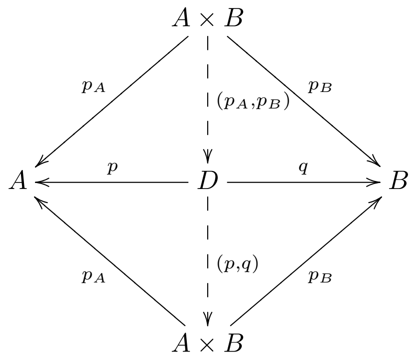
Moreover,

$$p \circ ((p_A, p_B) \circ (p, q)) = (p \circ (p_A, p_B)) \circ (p, q) = p_A \circ (p, q) = p.$$
$$q \circ ((p_A, p_B) \circ (p, q)) = (q \circ (p_A, p_B)) \circ (p, q) = p_B \circ (p, q) = q.$$

But this is exactly what we need in order to conclude that:

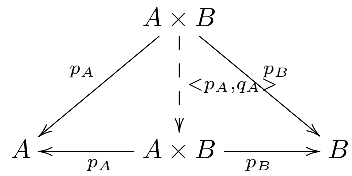
$$(p_A, p_B) \circ (p, q) =< p, q >= id_D.$$

We now repeat the above argument, replacing D with $A \times B$ and $A \times B$ with D.
We form the diagram:



We can then form the composite $(p, q) \circ (p_A, p_B) : A \times B \to A \times B$.

If we apply the product property of $A \times B$ to itself we get the following:



That is, there exists a unique morphism $< p_A, q_A >: A \times B \to A \times B$ such that:

$$p_A \circ < p_A, q_A >= p_A \text{ and } p_B \circ < p_A, q_A >= p_B.$$

Now, if we examine $id_{A\times B}$, then $id_{A\times B} : A \times B \to A \times B$, and,

$$p_A \circ id_{A\times B} = p_A$$
$$p_B \circ id_{A\times B} = p_B.$$

Hence we must have that $< p_A, q_A >= id_{A\times B}$.

Taking the morphism, $(p, q) \circ (p_A, p_B)$, then we want to show that it has the above properties, which would let us conclude that $< p_A, q_A >= id_{A\times B} = (p, q) \circ (p_A, p_B)$, to see this:

$$((p_A, p_B) \circ (p, q)) \circ p_A = (p_A, p_B) \circ ((p, q) \circ p_A) = (p_A, p_B) \circ p = p_A$$
$$((p_A, p_B) \circ (p, q)) \circ p_B = (p_A, p_B) \circ ((p, q) \circ p_B) = (p_A, p_B) \circ q = p_B$$

Hence:

$$id_{A\times B} = (p, q) \circ (p_A, p_B).$$

We therefore conclude that $D \cong A \times B$. $\square$

Example 0.2.96. In $\mathbb{SET}$ the product of two sets does in fact correspond to the cartesian product of the sets. To see this note, when forming the cartesian product of A and B, we take the set $A \times B$ and the morphisms left and right projection. If we have any set D and functions:

$$f : D \to A \text{ and } D :\to B$$

Then there is indeed a unique function $< f, g >: D \to A \times B$ such that:

$$p_A \circ < f, g >= f \text{ and } p_B \circ < f, g >= g.$$

(note that composition by p_A takes care of the left hand value, whilst composition by p_B ensures the right value.)

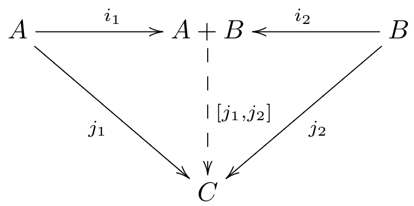
◇

Remark 0.2.97. In a poset $(P, \leq)$, the g.l.b (greatest lower bound) of a subset of $S \subseteq P$ is an element $s_0 \in S$, such that $s_0 \leq p$ for all $p \in P$ and moreover, if $s_1 \leq p$ for all $p \in P$, then $s_1 \leq s_0$.

Example 0.2.98. In a poset the product of two objects is the g.l.b. To see this, note that the existence of the two morphisms $p_a : A \times B \to A$ and $p_B : A \times B \to B$ ensures that $A \times B \leq A$ and $A \times B \leq B$ hence ensuring that $A \times B$ is a lower bound. The requirement that it be the greatest bound follows by supposing we have another bound (object with morphisms $p : C \to A$, $q : C \to B$) and then deriving the morphism $< p, q >: C \to A \times B$, which tells us that $C \leq A \times B$.

◇

Definition 0.2.99. *The dual notion of a product in a category is a coproduct, this is a triple* $(A + B, i_1 : A \to A + B, i_2 : B \to A + B)$ *such that for any object* C*, and pair of arrows* $j_1 : A \to C$ *and* $j_2 : B \to C$*, there exists a unique morphism* $[j_1, j_2]$ *such that the following diagram commutes:*



▷

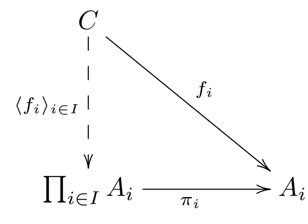
Example 0.2.100. In a poset the coproduct of two objects is the least upper bound of the two objects.

◇

Definition 0.2.101. *We can generalise the notion of a binary product to that of a product over an arbitrary collection. A product of a family* $(A_i)_{i \in I}$ *of objects in* $\mathcal{C}$*, indexed by a set* I *is a* $\mathcal{C}$ *object* $\prod_{i \in I} A_i$ *and a family of morphisms in* $\mathcal{C}$*,* $\{\pi_i : (\prod_{i \in I} A_i) \to A_i\}_{i \in I}$*, such that for all* $\mathcal{C}$ *objects and* $\mathcal{C}$ *morphisms* $\{f_i : C \to A_i\}_{i \in I}$*, there exists a unique arrow:*

$$\langle f_i \rangle_{i \in I} : C \to (\textstyle\prod_{i \in I} A_i)$$

Such that for all $i \in I$*, the following diagram commutes.*



▷

Example 0.2.102. Given a poset P, the product of a collection of objects is the g.l.b. of the collection seen as a subset of the poset. To see this note that the product is less than every element of the collection, but also, it is bigger than any other element of the poset with this property.

◇

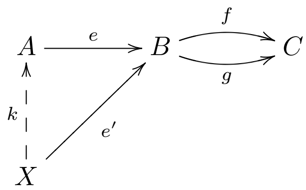
Definition 0.2.103. *The coproduct of a collection of objects is defined in the (fairly) obvious way, following on from the definition of the product of a collection of objects.*

▷

Definition 0.2.104. *A morphism $e : A \to B$ is an equalizer of two arrows $f : B \to C$ and $g : B \to C$ if:*
- $f \circ e = g \circ e$.
- $\forall\ e' : X \to B$ *such that* $f \circ e' = g \circ e'$ *There exists a unique morphisms* $k : X \to A$ *such that:*

$$e \circ k = e'.$$

This information is encoded in the diagram:



▷

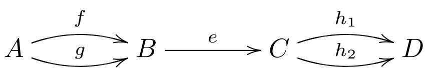
Remark 0.2.105. It is a general theme that in a poset, because all morphisms are already unique, whenever a construction requires the existence and uniqueness of a certain morphism, this then reduces to simply requiring the existence of the morphism.

Proposition 0.2.106. *In a poset, the equalizers are identities*

Proof. To see this note that we can take e' to be $id_B : B \to B$, then there exists some $k : B \to A$ such that $e \circ k = id_B$, but then we must have $A \leq B$ and $B \leq A$ yet from anti-symmetry this means that $A = B$, hence the equalizer $e : B \to B$ must be the unique morphism from B to B, i.e. $e = id_B$.
□

Proposition 0.2.107. *Every equalizer is a monomorphism.*

Proof. Suppose $e \circ f = e \circ g$, then we have the following:



But if we compose e and f, we end up with a morphism $e \circ f : A \to C$, where $(e \circ f) \circ h_1 = (e \circ f) \circ h_2$ which then gives us a unique morphism $k : A \to B$ such that $e \circ k = e \circ f$ but then upon the assumption that $e \circ f = e \circ g$, we have that $k = f = g$. □

Definition 0.2.108. *In $\mathbb{SET}$, a relation on a set X, is a subset $R \subseteq X \times X$. We then say that $x_1 \sim x_2$ iff $< x_1, x_2 > \in R$.*

▷

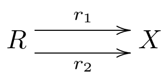
Definition 0.2.109. *An equivalence relation on a set X, is a relation $\sim$, such that satisfies the following properties;*
- *Reflexivity - $x \sim x$, for all $x \in X$.*
- *Symmetry - $x \sim y \implies y \sim x$, for all $x, y \in X$*
- *Transitivity - $x \sim y$ & $y \sim z \implies x \sim z$, for all $x, y, z \in X$.*

▷

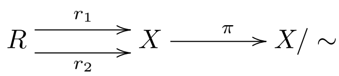
Remark 0.2.110. In $\mathbb{SET}$, given a set X, and an equivalence relation $\sim$ on X, then we can examine the two projection functions; $r_1 : R \to X$ and $r_2 : R \to X$, defined by $r_1(< a, b >) = a$ and $r_2(< a, b >) = b$

Proposition 0.2.111. *The coequalizer of these two projection functions is the quotient set $X/\sim$.*

Proof. We start with the two parallel functions:



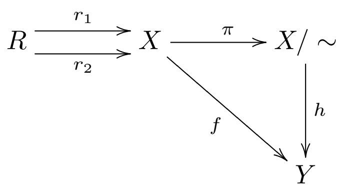
Then the claim is that the cannonical projection $\pi : X \to X/\sim$ is a coequlizer of r_1 and r_2. To see this we throw this function into the diagram:



We can deduce that $\pi \circ r_1 = \pi \circ r_2$, to see this:

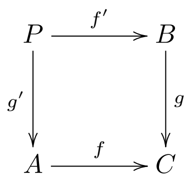
$$\pi(r_1(< a, b >)) = \pi(a) = [a] = [b] = \pi(b) = \pi(r_2(< a, b >)$$

We can say that $[a] = [b]$ because we know that $< a, b >$ is in R. Now suppose that we have a function $f : X \to Y$, where $f \circ r_1 = f \circ r_2$, then this condition tells us that $a \sim b \implies f(a) = f(b)$. Let us see everything we need for a coequalizer:

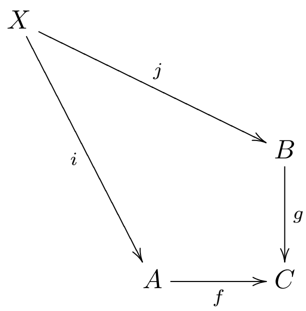


Then we require a unique function $h : X/\sim \to Y$, such that $h \circ \pi = f$. However we already know that f respects the equivalence relation, and therefore $h([a]) = f(a)$ is well defined, and furthermore, it is the unique such function. □

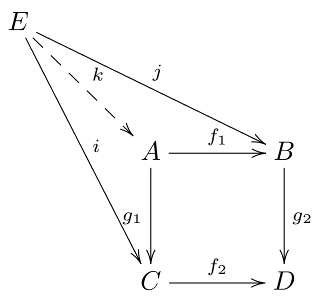
Definition 0.2.112. *A pullback of a pair of arrows $f : A \to C$ and $g : B \to C$ is an object P and a pair of arrows $f' : P \to A$ and $g' : P \to B$ such that $f \circ g' = g \circ f'$, this is captured in the diagram:*



We also require that for any other morphisms, $i : X \to A$ and $j : X \to B$ such that $f \circ i = g \circ j$:

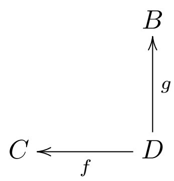


There exists a unique morphism $k : X \to P$ such that $i = g' \circ k$ and $j = f' \circ k$ i.e.

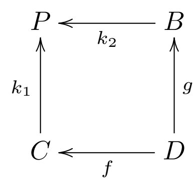


▷

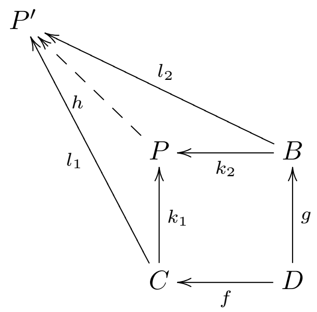
Definition 0.2.113. *The dual notion to a pullback is called a pushout. It is the set up that for two morphisms of the form:*



Supplies an object P and two morphisms $k_1 : C \to P$, $k_2 : B \to P$, such that $k_2 \circ g = k_1 \circ f$.



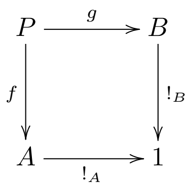
Furthermore, for other pair of morphisms $l_1 : C \to P'$, $l_2 : B \to P'$with this property, we have a unique morphism $h : P \to P'$ such that $h \circ k_2 = l_2$ and $h \circ k_1 = l_1$.



▷

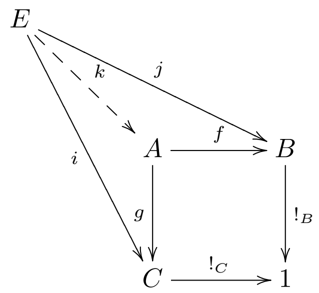
Remark 0.2.114. There are some interesting relations between the various notions that we have just defined, we explore these relations in the following two propositions and one theorem.

Proposition 0.2.115. *In a category $\mathcal{C}$, with a terminal object* 1*, if:*

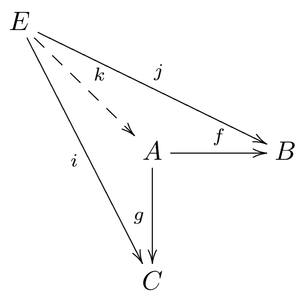


Is a pullback, then P is a product of A and B, where the morphisms f and g serve as the projection morphisms.

Proof. If the diagram is a pullback, then for any triple $(E, i : E \to C, j : E \to B)$ such that $!_C \circ i$ and $!_B \circ j$, we have the following:



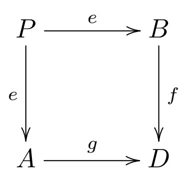
As 1 is a terminal object there is a unique morphism $!_E : E \to 1$ however we see that we can take two routes from E to 1. Firstly $!_C \circ i$ and secondly $!_B \circ j$, therefore these two most be equal. So the requirement that $!_C \circ i = !_B \circ j$ is true whenever we have the existence of the morphisms, i and j. Hence we simply end up with a product of B and C:



□

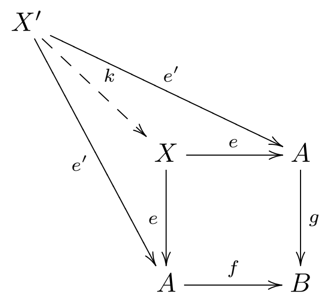
Remark 0.2.116. From the previous proposition, we can say that if a category has a terminal object, and every pair of morphisms has a pullback, then all objects have products.

Proposition 0.2.117. *In a category $\mathcal{C}$, if*

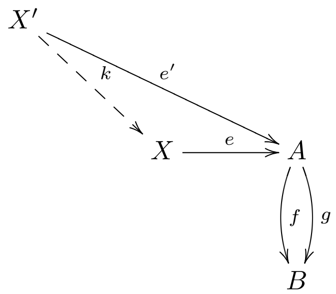


Is a pullback, then e is an equalizer of f and g.

Proof. If the diagram is a pullback, then for any morphism $e' : X' \to A$ such that $g \circ e' = f \circ e'$ we get a pullback:



But we can then rearrange this diagram by bringing the two A's together, to get:



Which is simply the equalizer of f and g.

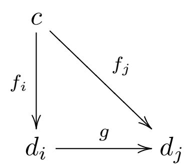
□

Remark 0.2.118. It might not have escaped the notice of the reader that the last few definitions (product, pullback, equalizer) had a very similar form, we described a certain class of morphisms, and then selected an object from this class that had some unique factoring properties. We explore this further now.

Definition 0.2.119. *A diagram in $\mathcal{C}$ is a directed graph in $\mathcal{C}$.*

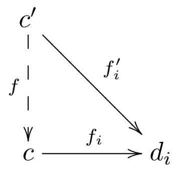
▷

Definition 0.2.120. *A cone for a diagram D is a $\mathcal{C}$ object C, and a collection of morphisms $f_i : c \to d_i$ (one for each object d_i in D) such that for every morphism $g : d_i \to d_j$ in the diagram, we have:*



▷

Definition 0.2.121. *A limit for a diagram D is a D-cone $\{f_i : c \to d_i\}$, such that for any other D-cone $\{f'_i : c' \to d_i\}$, there exists a unique morphism $f : c' \to c$ with:*



For every object d_i in D.

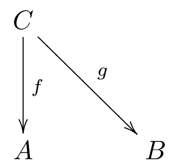
▷

Remark 0.2.122. A limit for a diagram D is unique up to isomorphism, the unique isomorphism is the morphism $f : c \to c'$

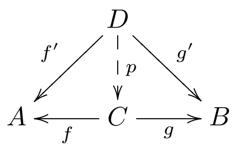
Example 0.2.123. Given two objects in a category $\mathcal{C}$, we can form the diagram consisting of just the two objects:

$$A \qquad\qquad B$$

Then a D-cone is an object $\mathcal{C}$, with two arrows f and g of the form:



And a limiting cone can be seen to be:



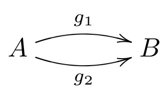
Which is a product of A and B.

◇

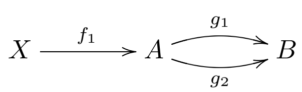
Example 0.2.124. Let D be the empty diagram in a category $\mathcal{C}$, then a cone for D is simply an object and a limiting cone is an object such that for any other object, there exists a unique morphism to the limit. Hence this corresponds to a terminal object.

◇

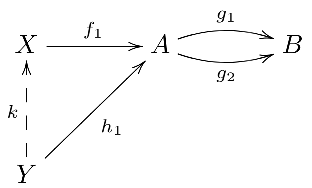
Example 0.2.125. Let D be the diagram:



Then a limit for this diagram is an equalizer. To see this first note that a cone for this diagram is an object X of $\mathcal{C}$, and a pair of morphisms $f_1 : X \to A$, $f_2 : X \to B$ such that $g_1 \circ f_1 = f_2$ and $g_2 \circ f_1 = f_2$. However we can see that $g_1 \circ f_1 = f_2 = g_2 \circ f_1$ so that f_2 is determined by f_1 hence, a cone for this diagram is actually an object X of $\mathcal{C}$ and a morphism $f_2 : X \to A$ such that $f_2 \circ g_1 = f_2 \circ g_2$:



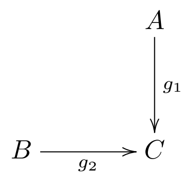
And then a limit is a cone $(X, f_1 : X \to A)$, such that for any other cone $(Y, h : Y \to A)$ there exists a unique arrow $k : Y \to$, with $f_1 \circ k = h$.



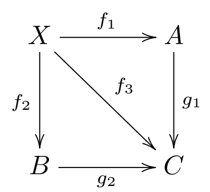
Hence we see that a limit for this diagram is an equalizer.

◇

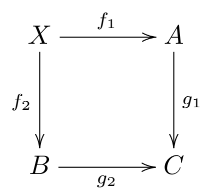
Example 0.2.126. Let D be the diagram:



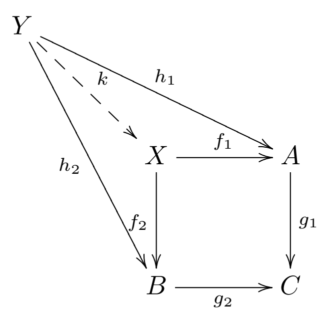
Then a cone for this diagram is a C object X and three morphisms $f_1 : X \to A$, $f_2 : X \to B$, $f_3 : X \to C$, such that $g_1 \circ f_1 = f_3 = g_2 \circ f_2$.



Hence we see that f_3 is determined by f_1 and f_2, hence a cone is a $\mathcal{C}$ object X and two morphisms $f_1 : X \to A$, $f_2 : X \to B$ such that $g_1 \circ f_1 = g_2 \circ f_2$.



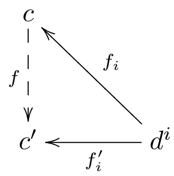
Then a limit for this diagram is a cone $(X, f_1 : X \to A, f_2 : X \to B)$ such that for any other cone $(Y, h_1 : Y \to A, h_2 : Y \to B)$ there exists a unique morphism $k : X \to Y$, where $h_1 = f_1 \circ k$ and $h_2 = f_2 \circ k$.



Hence we see that a limit for this diagram is a pullback.

◇

Definition 0.2.127. *By duality, we define a co-cone $\{f_i : d_i \to c\}$ for a diagram D to be an object $\mathcal{C}$ and a morphism $f_i : d_i \to c$ for every object d_i of D. A co-limit is then a co-cone such that for any other co-cone $\{f_i : d_i \to c'\}$, there exists a unique morphism $f : c \to c'$ such that:*



commutes for all d_i in D.

▷

All of the colimits below can be studied by simply looking at the dual of the limit of the diagram, but in order to provide some familiarity with co-limits as something to study on their own terms, we will work through a few examples.

Example 0.2.128. The co-limit for the empty diagram is an initial object. As mentioned above, we can deduce this from the fact that the limit for the empty diagram is a terminal object, and initial objects are dual to terminal objects. However let us work through it explicitly:

A Co-Cone for this diagram, is a $\mathcal{C}$ object, A, with a collection of morphisms $f_i : D_i \to A$, one for each object D_i, in the diagram, but as the diagram above is empty, there are no morphisms, hence a co-limit for this diagram is simply a $\mathcal{C}$, object A. Therefore a co-limit is a cone (A), such that for any other cone (B), there exists a unique morphism $f : A \to B$. Therefore we see that, a limit for this diagram is an initial object in the category.

◇

Example 0.2.129. The colimit of the following diagram is a coproduct:

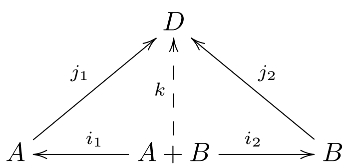
$$A \qquad\qquad B$$

We know that we have an object (which we will call $A + B$) and two morphisms, $i_1 : A \to A + B$ and $i_2 : B \to +B$. Then a colimit for this diagram is a cone $(A + B, i_1 : A \to A + B, i_2 : B \to A + B)$:

$$A \xleftarrow{i_1} A + B \xrightarrow{i_2} B$$

Such that for any other cone $(D, j_1 : A \to D, j_2 : B \to D)$, there exists a unique morphism
$k : A + B \to D$ such that:

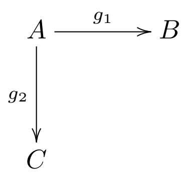
$$k \circ i_1 = j_1 \text{ and } k \circ i_2 = j_2.$$



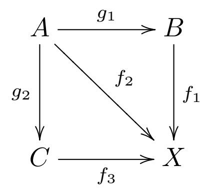
Hence we see that a colimit for this diagram is indeed a coproduct of A and B.

◇

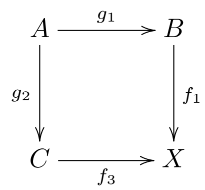
Example 0.2.130. The colimit of the following diagram is a pushout:



A co-cone for this diagram is then an object X and three morphisms $f_1 : B \to X$, $f_2 : A \to X$, $f_3 : C \to X$ such that $g_1 \circ f_1 = f_2 = g_3 \circ f_3$. In a diagram:

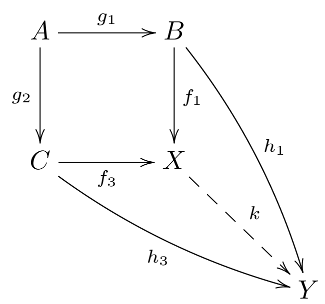


So we see that f_2 is determined by f_1 and f_3, hence a co-cone for this diagram is $\mathcal{C}$ object X, and two morphisms $f_1 : B \to X$, $f_3 : C \to X$ such that $f_1 \circ g_1 = f_3 \circ g_2$.



Then a co-limit for this diagram is a co-cone $(X, f_1 : B \to X, f_3 : C \to X)$ such that for any other co-cone $(Y, h_1 : B \to Y, h_3 : C \to Y)$ there exists a unique morphism $k : X \to Y$ such that:

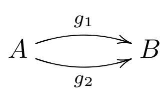
$$h_1 = k \circ g_1 \text{ and } h_3 = k \circ g_3.$$



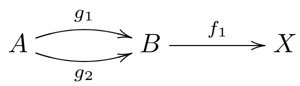
Hence we see that a colimit for this diagram is a pushout.

◇

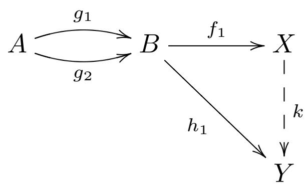
Example 0.2.131. The colimit of the following diagram is a coequalizer:



A cocone for this diagram is an object X and a pair of morphisms $f_1 : B \to X$ $f_2 : A \to X$ such that, $f_2 = f_1 \circ g_1$ and $f_2 = f_1 \circ g_2$, so that f_2 is determined by f_1, so we see that a cocone is an object X, and a morphism $f_1 : B \to X$ such that $f_1 \circ g_1$ and $f_1 \circ g_2$.



Then a co limit for this diagram is a co cone, $(X, f_1 : B \to X)$ such that for any other co cone $(Y, h_1 : B \to X)$, there exists a unique morphism $k : X \to Y$ such that $h_1 = k \circ f_1$.



Hence we see that a colimit for this diagram is a coequalizer.

◇

0.3 Functors and Natural Transformations

0.3.1 Functors

Definition 0.3.1. *A functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ comprises:*
1) A mapping F from $Obj\ \mathcal{C}$ to $Obj\ \mathcal{D}$
2) An assignment to each morphism $f : A \to B$ in $Mor\ \mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$
Such that:
- $F(fg) = F(f)F(g)$
- $F(e_A) = e_{F(A)}$

▷

Example 0.3.2. The identity functor. This functor maps from a category $\mathcal{C}$ back to $\mathcal{C}$, we denote it $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. This functor exists for every category. Note that, much like in the case of the identity function where each set has its own identity function, each category has its own identity functor.

To see that this is a functor, we note that we have a function $F : Obj\mathcal{C} \to Obj\mathcal{C}$ (the identity function $id_{Obj\mathcal{C}}$). Each morphism $f : A \to B$ is sent to the morphism $f : FA \to FB$, but as $fA = A$ and $fB = B$, this is just $f : A \to B$.

Clearly this functor respects composition, and identity morphisms are sent to identity morphisms.

◇

Our second functor is the forgetful functor, this functor takes a category of 'structured sets' (for example $\mathbb{GRP}$ or $\mathbb{TOP}$), and then sends each object to its underlying set, and each morphism to the underlying set function. Whilst it might seem like a fairly trivial functor, its importance will become apparent when we look at free constructions. Much like the identity functor, each category has a different forgetful functor (infact some categories have more than one), we shouldn't think that there is just one forgetful functor.

Example 0.3.3. In the case of $\mathbb{MON}$, the forgetful functor U, sends a monoid, $(M, \cdot, e_M)$ to the set M, and a monoid homomorphism $f : M_1 \to M_2$ to the underlying set function. We can see that $U(id_M : M \to M) = id_M$ and $U(f \circ g) = Uf \circ Ug$.

◇

Example 0.3.4. In the case of $\mathbb{TOP}$, the forgetful functor takes a topological space (X, V) and sends it to the set X. A continuous function $f : (X, V) \to (Y, W)$ between two topological spaces is sent to the function $f' : X \to Y$, which is the underlying function. (In fact the function f considered as a morphism in $\mathbb{TOP}$ isn't anything but a continuous set function, however as it sits in the category of topological spaces it is technically a completely different morphism from f', so we couldn't just say that f is sent to f, instead we need to make clear that f is sent to f'.)

◇

Example 0.3.5. The powerset functor. This is a functor $P : \mathbb{SET} \to \mathbb{SET}$ which sends each set to its powerset $P(A)$, and a function $f : A \to B$ to the function $P(f) : P(A) \to P(B)$ where;

$$P(f)(X) = f[X].$$

We can see that it sends sets to sets and that also it sends functions to functions in the correct way. The two remaining properties are the identity property and the composition property.

Identity:

$$p(id_A)(X) = id_A[X] = X.$$

Composition:
Given two functions $f : A \to B$, $g : B \to C$, in $\mathbb{SET}$, we want to show:

$$P(g \circ f) = P(g) \circ P(f).$$

Let X be a subset of A, then:

$$P(g \circ f)(X) = (g \circ f)[X] = g(f[X])) = P(g)(f[x]) = (P(g) \circ P(f))(X).$$

Hence we see that P is a functor.

◇

Remember that we can view a monoid as a category with one element, we can then ask what a functor between two monoids would look like.

Proposition 0.3.6. *A functor between two monoids is a monoid homomorphism.*

Proof. Recall that a monoid homomorphism is a function $\phi : M_1 \to M_2$ such that:

$$\phi(m_1 m_2) = \phi(m_1)\phi(m_2) \text{ and } \phi(e_1) = e_2,$$

If we let $F : M_1 \to M_2$ be a functor, between two morphisms considered as categories. As the identity morphism is the identity element, the requirement that the identity morphism of M_1 be sent to the identity morphism of M_2 is equivalent to sending the identity element of M_1 to the identity element of M_2.

Given two elements $m_1, m_2 \in M$, the product is $m_1 \circ m_2$, hence if F is a functor, then:

$$F(m_1 \cdot m_2) = F(m_1 \circ m_2) = F(m_1) \circ F(m_2) = F(m_1) \cdot F(m_2).$$

Hence a functor is a monoid homomorphism. □

Another familiar object that we can think of as a category is a poset, in a similar way we can ask what a functor between two posets looks like.

Example 0.3.7. A functor between two posets is a monotone function.

Recall that a monotone function $f : P \to Q$ is a function between partial orders $(P, \leq)$ and $(Q, \preceq)$, such that, $\forall p_1, p_2 \in P$:

$$p_1 \leq p_2 \implies f(p_1) \preceq f(p_2).$$

Suppose that we have a functor $f : P \to Q$, and suppose that $p_1 \leq p_2$, then this means that in the category P, there exists a morphism $x : p_1 \to p_2$, we can now apply the functor f to this morphism to obtain a morphism in Q:

$$fx : f(p_1) \to f(p_2).$$

But from this we can conclude that:

$$f(p_1) \preceq f(p_2).$$

So we see that a functor $f : P \to Q$ is a monotone function.

$\diamond$

So we see that the notion of a functor generalizes the notions of monoid homomorphisms and monotone functions. One important property of these two objects is that we are able to compose them, it turns out that we can in fact compose functors, and moreover with a few necessary restrictions, we can talk about a category of categories.

Proposition 0.3.8. *The composition of two functors is a functor.*

Proof. We first need to specify what exactly we mean by the 'composition of two functors'. We can think of a functor as a pair of related functions, one acting on objects, the other acting on morphisms.

Therefore, when we say the composition of two functors we are thinking of the assignment that acts on objects by taking the composition of the object functions, and acts on morphisms by taking the composition of the morphism functions.

Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors between categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, then we claim that the composition is a functor:

$$G \circ F : \mathcal{A} \to \mathcal{C}.$$

Firstly, objects in $\mathcal{A}$ are sent to objects in $\mathcal{B}$ by F, and then on to objects in $\mathcal{C}$ by G, therefore $G \circ F$ does indeed send objects of $\mathcal{A}$ to objects of $\mathcal{C}$. Also, a morphism $f : A_1 \to A_2$ is sent to the morphism:

$$Ff : f(A_1) \to f(A_2) \text{ in } \mathcal{A}.$$

Which is then sent to the morphism:

$$GFf : GF(A_1) \to GF(A_2) \text{ in } \mathcal{C}.$$

So $G \circ F$ displays the required behavior on objects and morphisms. It remains to verify that it respects identities and morphisms.

$$G \circ F(id_A) = G(id_{f(A)}) = id_{G \circ F(A)}.$$

Let $f : A_1 \to A_2$ and $g : A_2 \to A_3$ be two morphisms in $\mathcal{A}$, then:

$$G \circ F(g \circ f) = G(F(g \circ f)) = G(F(g) \circ F(f)) = G(F(g)) \circ G(F(f)).$$

Therefore we see that $G \circ F$ is a functor $\mathcal{A} \to \mathcal{C}$. $\square$

Proposition 0.3.9. *Functor composition is associative.*

Proof. When we compose functors we simply compose the two underlying functions, hence associativity of functor composition follows from associativity of function composition.

$\square$

Definition 0.3.10. *If we take the collection of all small categories, then this forms a category* $\mathbb{CAT}$*, where morphisms are functors between categories. We have already collected all the information that verifies that* $\mathbb{CAT}$ *is a category. The domains and codomain of a functor is clear, morphism composition is well defined from the above proposition, as is associativity, finally the identity functor that we defined above is the identity morphism.*

▷

Remark 0.3.11. Now that we have placed functors inside a category, it follows immediately that we have a notion of what it means for a functor to be an isomorphism, namely a functor is an isomorphism iff it is an isomorphism as a morphism.

Example 0.3.12. If we have a category $\mathcal{C}$ with all binary products, then for a given object A, we can define the functor $- \times A : A \to A$. It sends an object B in $\mathcal{C}$ to the product $B \times A$. A morphism $f : B \to C$ is sent to the morphism $f \times id_A : B \times A \to C \times A$.

Clearly this mapping sends objects of $\mathcal{C}$ to objects of $\mathcal{C}$, and morphisms of $\mathcal{C}$ to $\mathcal{C}$ in the required way, that is, given a morphism $f : B \to C$:

$$(- \times A)(f) = (f \times id_A) : B \times A \to C \times A.$$

We also need to check that $- \times A$ sends identities to identities and respects composition.
Take an identity morphism, $id_A : A \to A$, then $(- \times A)(id_A) = (id_A \times id_A)$, which is the identity morphism on the object $A \times A$.

Now take two morphisms $f : A \to B$, and $g : B \to C$ in $\mathcal{C}$, we need that:

$$(- \times A)(g \circ f) = (- \times A)(g) \circ (- \times A)(f).$$

To see this:

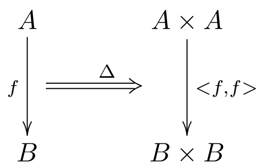
$$(- \times A)(g \circ f) = (g \circ f \times id_A) = (g \circ id_A) \circ (f \times id_A) = (- \times A)(g) \circ (- \times A)(f).$$

Hence we see that $(- \times A)$ is a functor.

◇

The following functor may also look unimportant, but it too will be seen to be fundamental when we come to look at adjoint functors and limits.

Example 0.3.13. For a category $\mathcal{C}$, the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ sends an object $A \in \mathcal{C}$ to the object $(A, A) \in \mathcal{C} \times \mathcal{C}$, and a morphism $(f : A \to B) \in \mathcal{C}$, to the morphism $(f, f) \in \mathcal{C} \times \mathcal{C}$.
Clearly this sends objects to objects, and a morphism:



We now need to check that Δ sends identities to identities:

$$\Delta(id_A) =< id_A, id_A >.$$

Which is indeed the identity on $A \times A$.

Now let $f : A \to B$, $g : B \to C$, be two morphisms in $\mathcal{C}$, we need that:

$$\Delta(g \circ f) = \Delta(g) \circ \Delta(f).$$

To see this:

$$\Delta(g \circ f) = < g \circ f, g \circ f > = < g, g > \circ < f, f > = \Delta(g) \circ \Delta(f).$$

Hence Δ is a functor.

◇

All the functors that we have introduced so far have been 'covariant', we now give the definition of what it means for a functor to be contravariant. We will continue referring to a 'covariant functor' as simply a 'functor' whereas a contravariant functor will always be called such.

Definition 0.3.14. *A contravariant functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ comprises:*
1) A mapping F from Obj $\mathcal{C}$ to Obj $\mathcal{D}$
2) An assignment to each morphism $f : A \to B$ in Mor $\mathcal{C}$, a morphism $F(f) : F(B) \to F(A)$, such that, for all pairs of composable morphisms $f : A \to B$, $g : B \to C$:

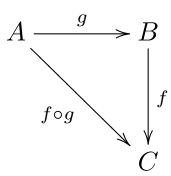
$$F(g \circ f) = F(f) \circ F(g).$$
$$F(e_A) = e_{F(A)}.$$

Notice that the image of a morphism acts in the opposite direction, and F swaps morphisms when composing them.

▷

Remark 0.3.15. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$, in fact some authors actually define a contravariant functor to be a covariant functor that has as its domain a dual category.

Definition 0.3.16. *For a locally small category $\mathcal{C}$, and an object A in $\mathcal{C}$, we define the hom functor $Hom(A, -) : \mathcal{C} \to \mathbb{SET}$ which sends a $\mathcal{C}$ object, B, to the hom set $Hom_{\mathcal{C}}(A, B)$, and a morphism $f : B \to C$ is sent to the function $Hom(A, f) : Hom(A, B) \to Hom(A, C)$, defined by $Hom(A, f)(g : A \to B) = f \circ g$*



▷

Proposition 0.3.17. *The hom functor as defined above is a functor.*

Proof. Firstly note that we have a mapping of objects in $\mathcal{C}$ to objects in $\mathbb{SET}$, and morphisms in $\mathcal{C}$ to functions in $\mathbb{SET}$, in a way that respects domains and codomains, it remains to check that the functor sends the identity morphism to the identity function, and respects morphisms.

$$Hom(A, id_B)(g : A \to B) = id_B \circ g = g.$$

We also need that:

$$Hom(A, g \circ f) = Hom(A, g) \circ Hom(A, f).$$

Then $Hom(A, g \circ f)(h : A \to B) = (g \circ f) \circ h$.
$Hom(A, f)(h : A \to B) = f \circ h$.
$Hom(A, g)(Hom(A, f)h) = g \circ (f \circ h)$. So we see that:

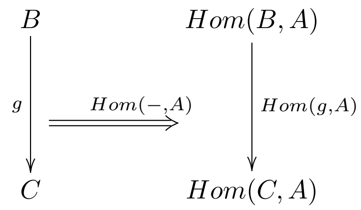
$$Hom(A, g \circ f) = Hom(A, g) \circ Hom(A, f).$$

And $Hom(A, -)$ is a functor. □

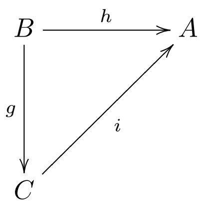
The natural next step is to attempt is to come up with a functor that sends an object A of $\mathcal{C}$ to the set $Hom(-, A)$. It turns out however that this is slightly less straightforward, to see this let us imagine that we assign the objects B and C to the sets $Hom(B, A)$ and $Hom(C, A)$, and then imagine that we are given a morphism $g : B \to C$ in $\mathcal{C}$, then we want to send this morphism to a morphism in $\mathbb{SET}$:

$$Hom(-, A)g : Hom(B, A) \to Hom(C, A).$$

That is in a diagram:



So now if we take two functions, h from $Hom(B, A)$ and i from $Hom(C, A)$, then we want to use g to send h to i:



But g is facing the wrong direction. What we can actually do is send i to h, by setting $h = i \circ g$, that is send an element of $Hom(C, A)$ to and element of $Hom(B, A)$. So we see that the functor $Hom(-, A)$ is actually contravariant, not covariant.

Definition 0.3.18. *We define the contravariant hom functor in a locally small category $\mathcal{C}$, for an object A, $Hom(-, A) : \mathcal{C} \to \mathbb{SET}$. It sends an object B of $\mathcal{C}$ to the set $Hom(B, A)$. A morphism $f : B \to C$ is sent to the function $Hom(f, A) : Hom(C, A) \to Hom(B, A)$, where:*

$$Hom(f, A)(g : B \to C) = f \circ g.$$

▷

Definition 0.3.19. *A bifunctor is a functor from a product category, that is a functor whose domain is the product of two categories.*

▷

Remark 0.3.20. If we fix one of the variables of the bifunctor then we end up with a standard functor. To see this, suppose that we have a bifunctor $K : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, and that we have selected an element A of $\mathcal{C}$, then we can form a functor $K_r : \mathcal{D} \to \mathcal{E}$, by sending an object B of $\mathcal{D}$ to $K(A, B)$, the functor properties of K_r follow from the functor properties of K. It turns out that a bifunctor is completely determined by its values under these restrictions, this relation is expressed in the following proposition.

Proposition 0.3.21. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, and for all objects $C \in C$ $D \in D$ let $L_D : \mathcal{C} \to \mathcal{E}$, $R_C : \mathcal{D} \to \mathcal{E}$ be functors such that $L_D(C) = R_C(D)$ for all C, D.*

Then, there exists a bifunctor $S : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that, for all C in $\mathcal{C}$ and D in $\mathcal{D}$:

$$S(-, D) = L_D \text{ and } S(C, -) = R_C.$$

Iff, for every pair of morphisms $f : C \to C'$, in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$, we have:

$$R_{C'} g \circ L_D f = L_{D'} f \circ R_C g.$$

At which point we say that $S(f, g)$ is equal to this value, that is:

$$S(f, g) = R_{C'} g \circ L_D f = L_{D'} f \circ R_C g.$$

Proof. We first assume that we have two such functors, $L_D : \mathcal{C} \to \mathcal{E}$, $R_C : \mathcal{D} \to \mathcal{E}$, where $R_{C'} g \circ L_D f = L_{D'} f \circ R_C g$, then we need to check that the following assignment is functorial:

$$S(A, B) = L_B A$$
$$S(f, g) = R_{C'} g \circ L_D f$$

We first check that it sends indentities to identities, for an object (C, D) in $\mathcal{C} \times \mathcal{D}$, the identity morphism is $< id_C, id_D >$, hence:

$$S(< id_C, id_D >) = R_C id_D \circ L_D id_C < id_c, id_D > \circ < id_C, id_D >=< id_C, id_D >.$$

So we see that S sends identities to identities, we now need to check that it respects composition. That is for all morphisms $f : C_1 \to C_2$, $h : C_2 \to C_3$ in $\mathcal{C}$, and $g : D_1 \to D_2$, $i : D_2 \to D_3$ in $\mathcal{D}$:

$$S(h \circ f, i \circ g) = S(h, i) \circ S(f, g)$$

We know that $S(h \circ f, i \circ g) = R_{C_3}(i \circ g) \circ L_{D_1}(h \circ f)$, hence:

$$S(h \circ f, i \circ g) = (R_{C_3} i \circ R_{C_3} g) \circ (L_{D_1} h \circ L_{D_1} f)$$

We now rearrange brackets:

$$S(h \circ f, i \circ g) = R_{C_3} i \circ (R_{C_3} g \circ L_{D_1} h) \circ L_{D_1} f$$

But now we can use the fact that, $(R_{C_3} g \circ L_{D_1} h) = L_{D_2} h \circ R_{C_2} g$ to get:

$$S(h \circ f, i \circ g) = R_{C_3} i \circ (L_{D_2} h \circ R_{C_2} g) \circ L_{D_1} f$$

Once again rearranging brackets leaves us with:

$$S(h \circ f, i \circ g) = (R_{C_3} i \circ L_{D_2} h) \circ (R_{C_2} g \circ L_{D_1} f)$$

And then putting in the definition of $S(h, i)$ and $S(f, g)$, we find that:

$$S(h \circ f, i \circ g) = S(h, i) \circ S(f, g)$$

So we see that S is a bifunctor.

To prove the opposite direction, we need to show that given functors $L_D : \mathcal{C} \to \mathcal{E}$, $R_C : \mathcal{D} \to \mathcal{E}$, such that, for all C, D:

$$L_D(C) = R_C(D).$$

and given a bifunctor $S : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that, for all C in $\mathcal{C}$ and D in $\mathcal{D}$:

$$S(-, D) = L_D \text{ and } S(C, -) = R_C.$$

Then for all morphisms, $f : C \to C'$, in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$, we have:

$$R_{C'}g \circ L_D f = L_{D'} f \circ R_C g.$$

We know that $R_{C'}g = S(C, g)$, and $L_D f = S(f, D)$, hence:

$$R_{C'}g \circ L_D f = S(C, g) \circ S(f, D).$$

But, $S(C, g) \circ S(f, D) = S(f, g) = S(f, D') \circ S(C', g)$, hence:

$$R_{C'}g \circ L_D f = S(f, D') \circ S(C', g).$$

But then using the fact that $S(f, D') = L_{D'} f$, and $S(C', g) = R_{C'}g$, we see that:

$$R_{C'}g \circ L_D f = L_{D'} f \circ R_{C'}g.$$

So we see that we have the required commutativity. $\square$

We are now able to present a few new constructions of categories, the last two of these are such that they require functors o be developed before they can be introduced, so we have waited until now to define them.

Example 0.3.22. The Hom functors, $Hom(A, -)$, and $Hom(-, B)$ together form a bifunctor $Hom(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbb{SET}$. We can see that $Hom(A, -)B = Hom(A, B) = Hom(-, B)A$ for all A in $\mathcal{C}^{op}$, B in$\mathcal{C}$. Hence from the last proposition, we require that for all morphisms, $f : C \to C'$ in $\mathcal{C}^{op}$, $g : D \to D'$ in $\mathcal{C}$:

$$Hom(-, C')g \circ Hom(D, -)f = Hom(D', -)f \circ Hom(C, -)g.$$

That is:

$$Hom(g, C') \circ Hom(D, f) = Hom(D', f) \circ Hom(C, g).$$

But:

$$Hom(g, C') \circ Hom(D, f)h = Hom(g, C')(f \circ h) = (f \circ h) \circ g$$

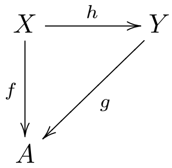
And:

$$Hom(D', f) \circ Hom(C, g)h = Hom(D', f)(h \circ g) = f \circ (h \circ g).$$

Hence from the above proposition $Hom(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbb{SET}$.

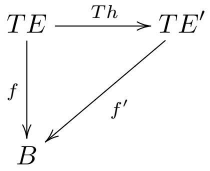
Recall that in an earlier chapter we defined the slice category.

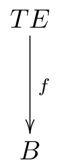
Definition 0.3.23. *Given a category $\mathcal{C}$, and an object A of $\mathcal{C}$, we can form the slice category $(\mathcal{C}, A)$, it is the category of all morphisms in $\mathcal{C}$, with codomain A. Then a morphism between these objects (morphisms, $f : X \to A$, and $g : Y \to A$), is an arrow $h : X \to Y$, such that $g \circ h = f$, this encapsulated in the following diagram:*



▷

Definition 0.3.24. *Given categories $\mathcal{C}$, $\mathcal{E}$, a functor $T : \mathcal{E} \to \mathcal{C}$, and an object $B \in \mathcal{C}$ the category $(T \downarrow B)$, has objects $< E, f >$, where $D \in Obj\mathcal{D}$, $f : TE \to B$, and an arrow $h :< E, f > \to < E', f' >$ is a morphism $h : E \to E'$ in $\mathcal{E}$, such that $f' \circ Th = f$.*



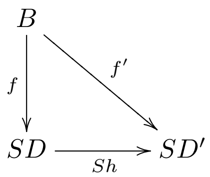


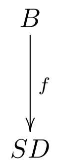
Composition of arrows is given by morphism composition in $\mathcal{E}$.

▷

We can think of this category as a generalization of a slice category, if we imagine taking $T : \mathcal{C} \to \mathcal{C}$ to be the identity functor, then $(T \downarrow B)$ is a category with objects, $< E, f >$ where E is an object in $\mathcal{C}$, and $f : TE \to B$. But then as $TE = E$, this means an object is a pair $< E, f >$ where $f : E \to B$. Then a morphism in this category $g :< f, C > \to < f', C' >$ is a morphism $h : E \to E'$ such that $g \circ h = f$. Therefore we see that this is the same as the slice category.

Definition 0.3.25. *Given categories $\mathcal{C}$, $\mathcal{D}$, a functor $S : \mathcal{D} \to \mathcal{C}$, and an object $B \in \mathcal{C}$ the category $(B \downarrow S)$ of objects S-under B, has objects $< D, f >$, where $D \in Obj\mathcal{D}$, $f : B \to SD$, and an arrow $h :< D, f > \to < D', f' >$ is a morphism $h : D \to D'$ in D, such that $f' = Sh \circ f$.*





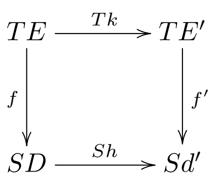
Composition of arrows is given by morphism composition in $\mathcal{D}$.

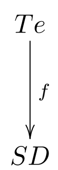
▷

Definition 0.3.26. *Given categories and functors:*

$$\mathcal{E} \xrightarrow{T} \mathcal{C} \xleftarrow{S} \mathcal{D}$$

We form the comma category $(T \downarrow S)$, which has objects (E, D, f) where E is an object in $\mathcal{E}$, D is an object in $\mathcal{D}$ and $f : TE \to SD$ is a morphism in $\mathcal{C}$. An arrow in $(T \downarrow S)$ $< E, D, f > \to < E', D', f' >$ is a pair of morphisms $< k, h >$ where $k : E \to E'$ and $h : D \to D'$ and $f' \circ Tk = Sh \circ f$





▷

Definition 0.3.27. *Each functor $F : \mathcal{C} \to \mathcal{D}$ between locally small categories $\mathcal{C}$ and $\mathcal{D}$ induces a function:*

$$F_{A,B} : Hom(A, B) \to Hom(FA, FB)$$

We say that the functor F is:
- *faithful if $F_{A,B}$ is injective.*
- *full if $F_{A,B}$ is surjective.*
- *fully faithful if $F_{A,B}$ is bijective.*

▷

Example 0.3.28. The forgetful functor $U : \mathbb{MON} \to \mathbb{SET}$ is faithful, that is each homomorphism is sent to a unique function, however it is not full, as there are some functions that are not also group homomorphisms.

Proposition 0.3.29. *If a functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, then $FA \cong FB \implies A \cong B$.*

Proof. We know that $FA \cong FB$, therefore there exists two morphisms $g : FA \to FB$ and $h : FB \to FA$ such that $g \circ f = id_{FB}$ and $f \circ g = id_{FA}$.

Since F is surjective, $\exists g'$ such that $F(g') = g$ and $\exists f'$ such that $F(f') = f$. Then:

$$F(g' \circ f') = F(g') \circ F(f') = g \circ f = id_{FB} = F(id_B).$$

Hence from faithfulness $g' \circ f' = id_B$. In the other direction:

$$F(f' \circ g') = F(f') \circ F(g') = f \circ g = id_{FA} = F(id_A).$$

And once again from faithfulness $f' \circ g' = id_A$. Therefore $A \cong B$. □

0.3.2 Natural Transformations

Now that we have turned a collection of categories into a category, we might ask whether it is possibly to go a step further and turn a collection of functors into a category 'natural transformations' allow us to do exactly this.

Definition 0.3.30. *Given two parallel functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\tau : F \to G$ is a collection of morphisms in $\mathcal{D}$:*

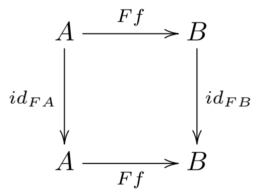
$$\tau = \{\tau_A : F(A) \to G(A) \mid A \in Obj\ \mathcal{C}\}.$$

One for each object in $\mathcal{C}$. Such that for each morphism $f : A \to B$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{t_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[t_B]{} & G(B) \end{array}$$

▷

Example 0.3.31. For any functor $F : C \to D$, there is always an 'identity natural transformation' id_F, which consists of all the identity morphisms in $\mathcal{C}$ under the map F, $id_F = \{id_{FA} \, | A \in \text{Obj } \mathcal{C}\}$, this gives us a collection of morphisms in $\mathcal{D}$, one for each object A in $\mathcal{C}$, finally the following diagram commutes:



That is:

$$id_{FB} \circ Ff = Ff = Ff \circ id_{FA}.$$

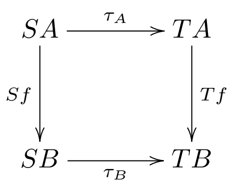
So that $id_F : F \to F$ is a natural transformation.

◇

Proposition 0.3.32. *Between two functors $S, T : \mathcal{C} \to P$, which map from an arbitrary category $\mathcal{C}$ to a poset P, there exists a natural transformation $\tau : S \to T$ iff $SA \leq TA$, $\forall A \in \mathcal{C}$.*

Proof. Recall that in a poset, $p_1 \leq p_2$ iff there exists a morphism $f : p_1 \to p_2$, for a natural transformation we need a morphism $f : SA \to TA$ for every object A in $\mathcal{C}$, we therefore see that there exist morphisms $\tau_A : SA \to TA$ iff $SA \leq TA$ for all objects A in $\mathcal{C}$.

What we have shown so far is that there is a collection of morphisms $\tau_A : SA \to TA$ iff $SA \leq TA$ for all objects A in $\mathcal{C}$, what we have not shown yet is that this collection is a natural transformation. Let $f : A \to B$ be a morphism in $\mathcal{C}$, then we need that the following diagram commutes:



That is

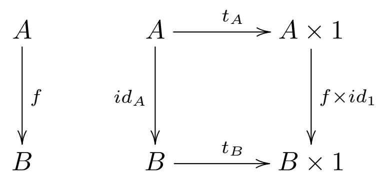
$$Tf \circ \tau_A = \tau_B \circ Sf.$$

But dom($Tf \circ \tau_A$) =dom ($\tau_B \circ Sf$) and cod($Tf \circ \tau_A$)=cod ($\tau_B \circ Sf$), and as we are working in a poset, where between any two objects there is at most one morphisms, this means that:

$$Tf \circ \tau_A = \tau_B \circ Sf.$$

□

Example 0.3.33. In the above section we had the functor, $(- \times A)$, which took an object B and sent it to $B \times A$. If we take the category $\mathbb{SET}$ then we claim that that there is a natural transformation, $t : id_{\mathbb{SET}} \to (- \times 1)$. Where 1 is the set $\{0\}$. Let us first look at what it would mean in order that t be a natural transformation. We would need a collection of morphisms $t_A : A \to A \times 1$, such that the following diagram commutes:



If we let $t_A(x) =< x, 0 >$, then:

$$(f \times id_1) \circ t_A(x) = (f \times id_1)(< x, 0 >) =< f(x), id_1(0) >=< f(x), 0 >= (t_B \circ f)(x).$$

So we see that t is a natural transformation.

◇

Example 0.3.34. We can also define a natural transformation $u : (- \times 1) \to id_{\mathbb{SET}}$, which acts by:

$$u_{A\times 1}(< x, 0 >) = x.$$

If we compose this natural transformation with the previous one in the following way: for each A, we look at $t_A \circ u_A$, and for each B, $u_B \circ t_B$, then we can see that we always get id_A and id_B, hence each component of the natural transformation t is an isomorphism. This leads us directly to the next definition

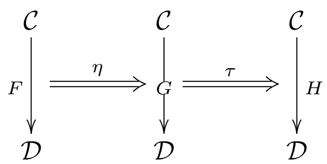
◇

Definition 0.3.35. *Given two parallel functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{D}$, a natural transformation $t : F \to G$ is said to be a natural isomorphism if each t_A is an isomorphism in $\mathcal{D}$.*

▷

We now develop the idea of taking functors between two fixed categories as objects in a new category.

Definition 0.3.36. *If we have two natural transformations $\eta : F \to G$, $\tau : G \to$, between functors F, G, and H, like so:*



Then we can combine the morphisms η_A and τ_A to get a morphism $(\tau \circ \eta)_A$, we define a new natural transformation $\tau \circ \eta$ to be the collection of all such morphisms.

▷

It should be clear that the new 'natural transformation' we defined above has the required collection of morphisms, however we don't know yet that $\tau \circ \eta$ respects morphisms in the required way, this is shown in the next proposition.

Proposition 0.3.37. *Given two natural transformations $\eta : F \to G$, $\tau : G \to H$, mapping between functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{C} \to \mathcal{D}$, then the collection:*

$$(\tau \circ \eta) = \{\eta_A \circ \tau_A \ |A \in Obj\ \mathcal{C}\}.$$

Is a natural transformation from F to H.

Proof. We need that for all morphisms $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle (\tau\circ\eta)_A}\downarrow & & \downarrow{\scriptstyle (\tau\circ\eta)_B} \\ HA & \xrightarrow{Hf} & HB \end{array}$$

That is:

$$(\tau \circ \eta)_B \circ Ff = Hf \circ (\tau \circ \eta)_A.$$

We have the commutativity of the following two diagrams:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ {\scriptstyle \tau_A}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ HA & \xrightarrow{Hf} & HB \end{array}$$

So:

$$(\tau \circ \eta)_B \circ Ff = (\tau_B \circ \eta_B) \circ Ff = \tau_B \circ (\eta_B \circ Ff) = \tau_B \circ (Gf \circ \eta_A) = (\tau_B \circ Gf) \circ \eta_A = (Hf \circ \tau_A) \circ \eta_A = Hf \circ (\tau_A \circ \eta_A) = Hf \circ (\tau \circ \eta)_A.$$

Hence we see that we have a collection of morphisms $\{\eta_A \circ \tau_A \ |A \in Obj\ \mathcal{C}\}$ in $\mathcal{D}$, one for each object in $\mathcal{C}$, with the required commutativity properties, hence $(\tau \circ \eta)$ is a natural transformation $F \to H$.
□

Definition 0.3.38. *Given two categories $\mathcal{C}$ and D, the functor category, denoted $\mathcal{D}^{\mathcal{C}}$ or $Funct(C, D)$ is the category with, objects, all the functors $F : C \to D$, and morphisms, natural transformations between functors.*

▷

Proposition 0.3.39. *Given categories $\mathcal{C}$ and $\mathcal{D}$. the functor category $\mathcal{D}^{\mathcal{C}}$ is also category*

Proof. We have already established a number of the required properties, we have a collection of objects, a collection of morphisms with domains and codomains, and above we established that we have a composition of morphisms, at the beginning of the section we introduced the identity natural transformation, this acts as the identity morphism, the only remaining property is that composition is associative, this follows from associativity in $\mathcal{D}$. □

Proposition 0.3.40. *A natural transformation $\tau : F \to G$, for functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{D}$ is a natural isomorphism iff it is an isomorphism in the category $\mathcal{D}^{\mathcal{C}}$.*

Proof. We begin by assuming that τ is a natural isomorphism, that is, all the components τ_A are isomorphisms in the category $\mathcal{D}$. Therefore each τ_A has an inverse, $(\tau_A)^{-1}$, if we take the collection of all such inverses:

$$\tau^{-1} = \{(\tau_A)^{-1} : GA \to FA \ |A \in \mathcal{C}\}.$$

We want to show that this collection τ^{-1} is a natural transformation. We have a collection of morphisms, one for each object A in $\mathcal{C}$. We also need that for every morphism $f : A \to B$, in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ {\scriptstyle (\tau^{-1})_A}\big\downarrow & & \big\downarrow{\scriptstyle (\tau^{-1})_B} \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

That is:

$$(\tau^{-1})_B \circ Gf = Ff \circ (\tau^{-1})_A.$$

To see that this is the case, we know that τ is a natural transformation $\tau : F \to G$ so that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \tau_A}\big\downarrow & & \big\downarrow{\scriptstyle \tau_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

i.e.

$$\tau_B \circ Ff = Gf \circ \tau_A.$$

Then multiplying on the left by $(\tau_B)^{-1}$ gives us:

$$(\tau_B)^{-1} \circ (\tau_B \circ Ff) = (\tau_B)^{-1} \circ (Gf \circ \tau_A).$$

Rearranging brackets:

$$((\tau_B)^{-1} \circ \tau_B) \circ Ff = ((\tau_B)^{-1} \circ Gf) \circ \tau_A.$$

Using the identity: $((\tau_B)^{-1} \circ \tau_B) = id_B$, we get:

$$id_B \circ Ff = ((\tau_B)^{-1} \circ Gf) \circ \tau_A.$$

The using the identity property:

$$Ff = ((\tau_B)^{-1} \circ Gf) \circ \tau_A.$$

We then multiply on the left by $(\tau_A)^{-1}$, and cancel to get:

$$Ff \circ (\tau_A)^{-1} = (\tau_B)^{-1} \circ Gf$$

Giving us the required result. Therefore τ^{-1} is a natural transformation $G \to F$. We now note that $\tau^{-1} \circ \tau = id_F$, and $\tau \circ \tau^{-1} = id_G$. Therefore τ which was assumed to be a natural isomorphism has been shown to be a natural transformation with an inverse, and hence an isomorphism in the category $\mathcal{D}^{\mathcal{C}}$.

It still remains to show that an isomorphism in $\mathcal{D}^{\mathcal{C}}$ is a natural isomorphism in the original sense.

Assume that $\tau : F \to G$, is an isomorphism in $\mathcal{D}^{\mathcal{C}}$, hence there exists a natural transformation τ^{-1} such that:

$$\tau^{-1} \circ \tau = id_F.$$
$$\tau \circ \tau^{-1} = id_G.$$

If we take a $\tau_A : FA \to GA$, then $(\tau^{-1})_A : GA \to FA$, and moreover $(\tau^{-1})_A \circ \tau_A = id_A$ and $\tau_A \circ (\tau^{-1})_A = id_A$, hence τ_A is an isomorphism. Therefore τ is a natural isomorphism.
□

There are three main ways that you can create a new natural transformation. The one explained above is important for the fact that it allows us to define functor categories, for this reason it will be denoted $\circ$. The second way also allows us to combine two natural transformations to get a third. The final way allows us to compose a natural transformation with a functor in order to obtain a new natural transformation.

We now work through the second method to obtain a new natural transformation from two old ones. Suppose that we have functors and natural transformations as follows:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} & \xrightarrow{J} & \mathcal{E} \\ & \Big\Downarrow \tau & & \Big\Downarrow \eta & \\ \mathcal{C} & \xrightarrow[G]{} & \mathcal{D} & \xrightarrow[K]{} & \mathcal{E} \end{array}$$

Then we can form a natural transformation $\eta \circ \tau : JF \to KG$. We call this 'horizontal composition' of natural transformations as opposed to vertical composition which we have already defined, if we ever just say 'composition' then it will refer to vertical composition.

Begin by choosing an object A in $\mathcal{C}$, then we can use the natural transformation τ to get a morphism τ_A in $\mathcal{D}$, $\tau_A : Fa \to GA$,

$$FA \xrightarrow{\tau_A} GA$$

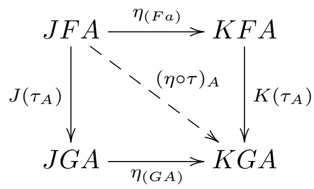
However, we also know that η is a natural transformation, and hence, as τ_A is a morphism in $\mathcal{D}$, we must have the commutativity of the following diagram:

$$\begin{array}{ccc} JFA & \xrightarrow{\eta_{(Fa)}} & KFA \\ {\scriptstyle J(\tau_A)}\Big\downarrow & & \Big\downarrow{\scriptstyle K(\tau_A)} \\ JGA & \xrightarrow[\eta_{(GA)}]{} & KGA \end{array}$$

That is:

$$K(\tau_A) \circ \eta_{(Fa)} = \eta_{(GA)} \circ J(\tau_A).$$

We now define $(\eta \circ \tau)_A$ to be the morphism that these two are equal to:



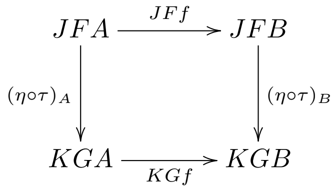
And at this point we claim that $\eta \circ \tau$ is a natural transformation, $JF \to KG$.

Proposition 0.3.41. *Given two natural transformations η and τ, as above, the collection:*

$$\{(\eta \circ \tau)_A \mid A \in \mathit{Obj}\, \mathcal{C}\}.$$

Is a natural transformation, $JF \to KG$.

Proof. We have a collection of morphisms in $\mathcal{E}$, one for each A in $\mathcal{C}$. It still remains to show that for all morphisms $f : A \to B$, in $\mathcal{C}$, the following diagram commutes:



That is:

$$(\eta \circ \tau)_B \circ JFf = KGf \circ (\eta \circ \tau)_A.$$

Then, putting in the definition, $(\eta \circ \tau)_B = \eta_{(GB)} \circ J(\tau_B)$ we find that we actually need:

$$(\eta_{(GB)} \circ J(\tau_B)) \circ JFf = KGf \circ (\eta \circ \tau)_A.$$

Putting in the definition $(\eta \circ \tau)_A = K(\tau_A) \circ \eta_{(Fa)}$ we get the requirement:

$$(\eta_{(GB)} \circ J(\tau_B)) \circ JFf = KGf \circ (K(\tau_A) \circ \eta_{(Fa)}).$$

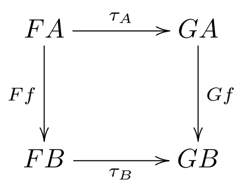
Then rearranging brackets:

$$\eta_{(GB)} \circ (J(\tau_B) \circ JFf) = (KGf \circ K(\tau_A)) \circ \eta_{(Fa)}.$$

We can now use the functor property of J and K to get to:

$$\eta_{(GB)} \circ (J(\tau_B \circ Ff)) = (K(Gf \circ \tau_A)) \circ \eta_{(Fa)}.$$

We now recall that τ is a natural transformation and that the following diagram commutes:



That is, $Gf \circ \tau_A = \tau_B \circ Ff$, we can then plug this in to get:

$$\eta_{(GB)} \circ (J(Gf \circ \tau_A)) = (K(Gf \circ \tau_A)) \circ \eta_{(Fa)}.$$

We now note that $Gf \circ \tau_A$ is a morphism in $\mathcal{D}$, and hence as η is a natural transformation $\eta : J \to K$, the following diagram must commute:

$$\begin{array}{ccc} JFA & \xrightarrow{\eta_{FA}} & KFA \\ {\scriptstyle J(Gf\circ\tau_A)}\downarrow & & \downarrow{\scriptstyle K(Gf\circ\tau_A)} \\ JGB & \xrightarrow[\eta_{GB}]{} & KGB \end{array}$$

Giving us the required result.

□

We now work throught the final method to obtain new natural transformations, this process is sometimes referred to as whiskering. Suppose that we have a natural transformation $\eta : F \to G$, and two additional functors $I : \mathcal{D} \to \mathcal{E}$, $H : \mathcal{B} \to \mathcal{C}$:

$$\begin{array}{ccc} \mathcal{B} & & \\ \downarrow{\scriptstyle H} & & \\ \mathcal{C} & & \mathcal{C} \\ {\scriptstyle F}\downarrow & \xRightarrow{\eta} & \downarrow{\scriptstyle G} \\ \mathcal{D} & & \mathcal{D} \\ & & \downarrow{\scriptstyle I} \\ & & \mathcal{E} \end{array}$$

Then we can compose the functors H and I with the natural transformation to get another natural transformation.

If we start with H, then we get a natural transformation $\eta H : FH \to GH$, and if we take I instead, we get a natural transformation $I\eta : IF \to IG$.

Remark 0.3.42. It's not obvious that we can form a natural transformation in this way, however we can see fairly quickly that it's definitely possible. A natural transformation $\tau : F \to G$, is a collection of morphism in $\mathcal{D}$, that map $FA \to GA$, where A is an object in $\mathcal{C}$.
A natural transformation $\tau : FH \to GH$ would once again be a collection of morphisms in $\mathcal{D}$, but this time mapping from $FHB \to GHB$, we can think of this as $F(HB) \to G(HB)$.
But we know that $HB = A$ for some A in $\mathcal{C}$. So we might suppose that we can take each object B in $\mathcal{B}$, send it to HB in $\mathcal{C}$, and then take the morphism $\eta_{(HB)}$, it turns out that this collection forms a natural transformation. A similar process works for composing natural transformations with functors on the right, we give the formal explanation below.

Proposition 0.3.43. *If we have a natural transformation $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$, and a functor $H : \mathcal{B} \to \mathcal{C}$ then the collection:*

$$\{\eta_{(HB)} : FHB \to GHB \mid B \in Obj\, \mathcal{B}\ \}.$$

Is a natural transformation between $\eta H : FH \to GH$.

Proof. We saw above that we have a collection of morphisms mapping between appropriate object in $\mathcal{D}$, we still need to show that for any morphism $f : A \to B$ in $\mathcal{B}$ the following diagram commutes:

$$\begin{array}{ccc} FHA & \xrightarrow{(\eta H)_A} & GHA \\ {\scriptstyle FHf}\downarrow & & \downarrow{\scriptstyle GHf} \\ FHB & \xrightarrow[(\eta H)_B]{} & GHB \end{array}$$

That is :

$$GHf \circ (\eta H)_A = (\eta H)_B \circ FHf.$$

Putting in the definition of ηH gives us:

$$GHf \circ \eta_{HA} = \eta_{HB} \circ FHf.$$

But at this point we note that $Hf : HA \to HB$ is a morphism in $\mathcal{C}$, calling this morphism $g : A' \to B'$, where $A' = HA$, and $B' = HB$, gives us:

$$Gg \circ \eta_{A'} = \eta_{B'} \circ Fg.$$

But we know that this is true because η is a natural transformation, $F \to G$. $\square$

Proposition 0.3.44. *If we have a natural transformation $\eta : F \to G$, between functors $F, G : \mathcal{C} \to \mathcal{D}$, and another functor $I : \mathcal{D} \to \mathcal{E}$, then the collection:*

$$\{I(\eta_A) : IFC \to IGA \mid A \in \mathit{Obj}\ \mathcal{C}\ \}.$$

Is a natural transformation between $I\eta : IF \to IG$.

Proof. We need a collection of morphisms in $\mathcal{E}$ that map from IFA to IGA, and we need one morphisms per object A in $\mathcal{C}$, moreover we need that for every morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} IFA & \xrightarrow{(I\eta)_A} & IGA \\ {\scriptstyle IFf}\downarrow & & \downarrow{\scriptstyle IGf} \\ IFB & \xrightarrow[(I\eta)_B]{} & IGB \end{array}$$

We will first establish the existence of the required morphisms. We know that η is a natural transformation $F \to G$, i.e. it is a collection of morphisms $\{\eta_A : FA \to GA \mid A \in \text{Obj}\ \mathcal{C}\}$. If we compose each η_A with I, then we get a morphism in $\mathcal{E}$, $I(\eta_A) : IFA \to IGA$, but this is precisely what we are look for. We claim that the collection $I\eta = \{I\eta_A : IFA \to IGA \mid A \in \text{Obj}\ \mathcal{C}\}$ that we have demonstrated the existence of, has the required commutativity properties. From the above diagram we can see that we need:

$$IGf \circ (I\eta)_A = (I\eta)_B \circ IFf.$$

Putting in the definitions of $I\eta_A$ and $I\eta_B$, gives us:

$$IGf \circ I(\eta_A) = I(\eta_B) \circ IFf.$$

Then factoring out I gives us:

$$I(Gf \circ \eta_A) = I(\eta_B \circ Ff).$$

So we see that if $Gf \circ \eta_A = \eta_B \circ Ff$ we would have the required result, but this follows from the fact that η is a natural transformation $F \to G$. □

Now that we have seen these three ways to build a new natural transformation it might be fruitful to ask whether they are related in any way. Though it is beyond the scope of this project, I will say that horizontal and vertical composition of natural transformations are related by an identity, and through this help to define a 2-category. A 2-category can be thought of a a category where every hom set has the structure of a category. 2-categories and their generalization n-categories are an important field of study, though not something that I will go into here.

We developed the notion of isomorphic categories above, however there is a related notion which is more useful in practice, that of equivalent categories.

Definition 0.3.45. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if there exists another functor $G : \mathcal{D} \to \mathcal{C}$, and two natural isomorphisms $\eta : 1_{\mathcal{C}} \cong GF$, $\tau : 1_{\mathcal{D}} \cong FG$.*

▷

Proposition 0.3.46. *Every isomorphism of categories is an equivalence of categories.*

Proof. We begin by assuming that we have two categories $\mathcal{C}$ and $\mathcal{D}$, and that we have two functors that map between these categories:

$$F : \mathcal{C} \to \mathcal{D}$$
$$G : \mathcal{D} \to \mathcal{C}.$$

And we also assume that:

$$F \circ G = id_{\mathcal{C}}$$
$$G \circ F = id_{\mathcal{D}}.$$

In order to show that $\mathcal{C}$ and $\mathcal{D}$ are equivalent, all we need are natural isomorphisms:

$$\eta : 1_{\mathcal{C}} \cong GF, \tau : 1_{\mathcal{D}} \cong FG.$$

However we can simply take the identity natural transformations on the functors $id_{\mathcal{C}}$ and $id_{\mathcal{D}}$, to conclude that:

$$F \circ G \cong id_{\mathcal{C}}$$
$$G \circ F \cong id_{\mathcal{D}}.$$

Therefore we see that every isomorphism of categories is also an equivalence.

□

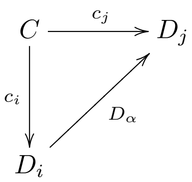
0.3.3 Limits

We now return to the notion of a limit, however, this time we will be using the language of functors. The reason for this is two fold, firstly functors are the natural language with which to talk about limits, secondly it will allow us to generalise limits. When we reuse a term you should take it that we are redefining it rather than stating an equivalent definition.

Definition 0.3.47. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories, a diagram of type $\mathcal{J}$ in $\mathcal{C}$ is a functor $D : \mathcal{J} \to \mathcal{C}$. J is said to be the index category.*

▷

Definition 0.3.48. *Given a diagram D, a cone for the diagram consists of an object of $\mathcal{C}$ and a collection of arrows in $\mathcal{C}$, $\{c_i : C \to D_i\}$, one for each object $i \in J$, where $D_i = D(i)$ such that $\forall \alpha : i \to j \in J$ we have:*

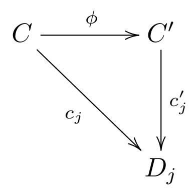


▷

Definition 0.3.49. *A morphism of cones:*

$$\phi :< c, c_i > \to < c', c'_i >$$

Is an arrow in $\mathcal{C}$, $\phi : c \to c'$ such that: $c'_i \circ \phi = c_j$, i.e. the following diagram commutes.



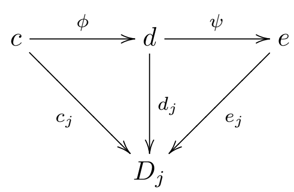
▷

Proposition 0.3.50. *Cones and morphisms of cones form a category, we denote this $Cone(D)$.*

Proof.
We have a well defined collection of objects and morphisms, moreover we have clearly defined domains and codomains for each morphism, the remaining properties are; composition, associativity, and identity.

Composition: Given two morphisms of cones; $\phi :< c, c_i > \to < d, d_i >$ and $\psi :< d, d_i > \to < e, e'_i >$, then composing this arrow does indeed give us an arrow:

$$\psi \circ \phi : c \to e.$$



Then,

$$(e_j \circ \psi) \circ \phi = d_j \circ \phi = c_j.$$

Associativity follows from associativity of morphisms in $\mathcal{C}$. For the identity cone morphism we take the identity morphism. $\square$

Definition 0.3.51. *A limit for a diagram $D : J \to C$ is a terminal object in $Cone(D)$*

$\triangleright$

Proposition 0.3.52. *Limits are unique up to isomorphism, and there is a unique isomorphism between isomorphic limits.*

Proof. This follows straightaway from the fact that limits are defined to be terminal objects for a particular category. We know that terminal objects are unique up to isomorphism and that between isomorphic terminal objects, there is a unique isomorphism. $\square$

Example 0.3.53. Let $\mathcal{J}$ be discrete category consisting of two objects:

$$A \qquad\qquad B$$

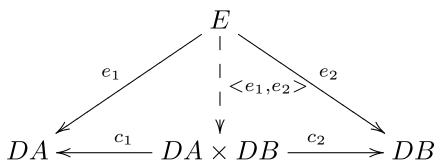
Then for a category $\mathcal{C}$, a diagram of type J is a functor $D : \mathcal{J} \to \mathcal{C}$. A cone for this diagram consists of a $\mathcal{C}$ object (which for the moment we will call $DA \times DB$), and a pair of morphisms $c_1 : DA \times DB \to DA$, $c_2 : DA \times DB \to DB$, giving us the following information:

$$DA \xleftarrow{c_1} DA \times DB \xrightarrow{c_2} DB$$

Then a limit for this diagram is a cone (let us suppose that the above cone is taken to be a limit) such that for any other cone, $(E, e_1 : B \to DA, e_2 : E \to DB)$, there exists a unique morphism $< e_1, e_2 >: E \to DA \times DB$ such that:

$$c_1 \circ < e_1, e_2 >= e_1 \text{ and } c_2 \circ < e_1, e_2 >= e_2.$$

So all of this can be encapsulated in the diagram:



Therefore, we can see that a limit for the diagram $D : \mathcal{J} \to \mathcal{C}$ is a product of the two objects DA and DB in $\mathcal{C}$, and that to take the product of any other objects in $\mathcal{C}$, we just need to select a different D, that picks them out instead.

$\diamond$

Example 0.3.54. If we take $\mathcal{J}$ to be the empty category, the category with no objects and no morphisms, then we can see that for a category $\mathcal{C}$, a diagram $D : \mathcal{J} \to \mathcal{C}$, is a functor that maps from the empty category to $\mathcal{C}$, and hence must be the empty functor. Therefore, a cone for this diagram is a $\mathcal{C}$ object A, and a collection of morphisms, one for each object in the index category, however as there are no objects in the index category, this reduces to just requiring that a cone be an object A of $\mathcal{C}$.

A limit for this diagram is then a cone (object A in $\mathcal{C}$), such that, for any other cone (object B in $\mathcal{C}$), there exists a unique morphism $f : A \to B$, such that f commutes with all the morphism in the cone. However as there are no morphism in the cone, this once again reduces and we are left with:

A limit for this diagram is an object A of $\mathcal{C}$, such that for any other object B in $\mathcal{C}$, there exists a unique morphism $f : A \to B$. Hence we see that a limit for this diagram is just a terminal object in $\mathcal{C}$.

◇

Remark 0.3.55. All of the above can be dualized to define co limits.

Theorem 0.3.56. *Limit Theorem*

Given a category $\mathcal{C}$ and a small index category $\mathcal{J}$, if all parallel morphisms in $\mathcal{C}$ have an equalizer, and $\mathcal{C}$ has all products indexed by the sets Obj $\mathcal{J}$, Mor $\mathcal{J}$ then $\mathcal{C}$ has a limit for every diagram $F : \mathcal{J} \to \mathcal{C}$.

Proof.
Clearly the left hand side implies the right hand side as a small product is a type of small limit, and similarly, equalizers are limits. The other direction is much less trivial.

We first take the two products $\prod_{i \in Obj\mathcal{J}} Fi$, and $\prod_{u \in Mor\mathcal{J}} F(cod(u))$.

$$\prod_u F(cod(u)) \qquad\qquad \prod_i Fi$$

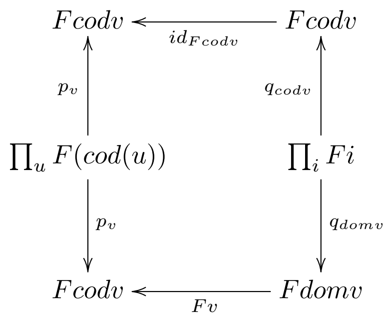
Then, for a given $v : j \to k$, take the projections, $p_v : \prod_u F(cod(u)) \to Fcod(v)$, giving us:

$$\begin{array}{ccc}
Fcod(v) & & \\
\uparrow{\scriptstyle p_v} & & \\
\prod_u F(cod(u)) & & \prod_i Fi \\
\downarrow{\scriptstyle p_v} & & \\
Fcodv & &
\end{array}$$

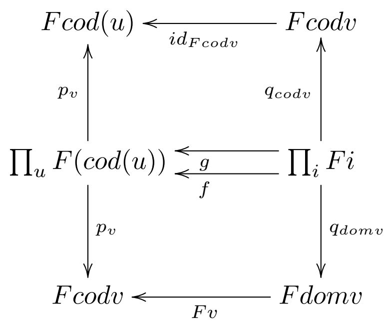
We now take the projections $q_{codv} : \prod_i Fi \to Fcodv$ and $q_{domv} : \prod_i Fi \to Fdomv$, giving us:

$$\begin{array}{ccc}
Fcod(v) & & Fcodv \\
\uparrow{\scriptstyle p_v} & & \uparrow{\scriptstyle q_{codv}} \\
\prod_u F(cod(u)) & & \prod_i Fi \\
\downarrow{\scriptstyle p_v} & & \downarrow{\scriptstyle q_{domv}} \\
Fcodv & & Fdomv
\end{array}$$

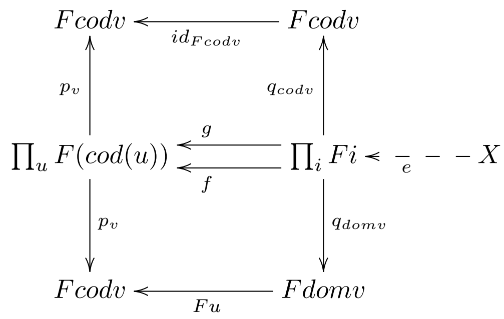
But we know that $Fu : Fdomv \to Fcodv$, and that $id_{Fcodv} : Fcodv \to Fcodv$, hence we have the existence of all the morphisms in the following diagram:



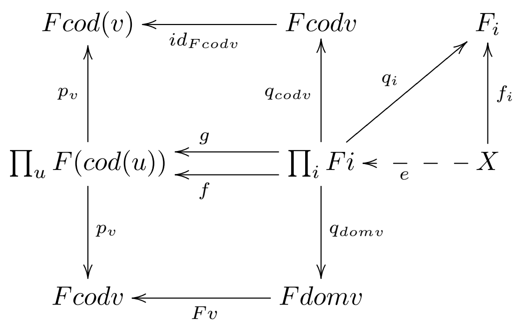
But as $\prod_u F(cod(u))$ is a product, we must have two unique morphisms $f, g : \prod_i Fi \to \prod_u F(cod(u))$, where g is the unique morphism making the upper square commute, and f is the unique morphism making the lower square commute:



We can now take the equalizer of the two parallel morphisms f and g, this gives us the following diagram:

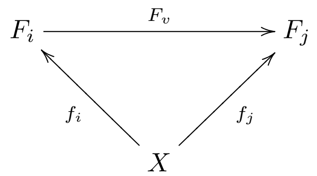


We now compose this e with the projections q_i to get morphisms $f_i : X \to F_i$. Like so:



We claim that $< X, f_i >$ is a limiting cone for F. To see this we need to show first that this is a cone, and secondly, for any other cone $< Y, g_i >$ there exists a unique morphism $h : Y \to X$ such that $g_i = f_i \circ h$.

We already know that X is an object of $\mathcal{C}$ and that we have a collection of morphism $f_i :\to F_i$ in $\mathcal{C}$ one for each object i in $\mathcal{J}$, so all that remains is to show that given $v : i \to j$ in $\mathcal{J}$ the following diagram commutes:



That is, $F_v \circ f_i = f_j$, but we know that, $f_i = q_i \circ e$, and $f_j = q_j \circ e$, so we actually want to show that:

$$F_v \circ q_i \circ e = q_j \circ e.$$

But $i = domv$, and $j = codv$ hence we want:

$$F_v \circ q_{domv} \circ e = q_{codv} \circ e.$$

But then from the main commutative diagram we have:

$$(Fv \circ q_{domv}) \circ e = (p_v \circ f) \circ e = p_v \circ (f \circ e) = p_v \circ (g \circ e) = (p_v \circ g) \circ e = q_{codv} \circ e.$$

As required. Therefore the collection $< X, f_i >$ is indeed a cone for the diagram F. We now need to check that it is a limiting cone, that is, for any other cone $< Y, g_i >$ there exists a unique morphism $h : Y \to X$ such that $g_i = f_i \circ h$.

If we assume the existence of the cone $< Y, g_i >$, then we have a collection of maps $Y \to F_i$, but this gives us a unique morphism $k : Y \to \prod_i Fi$ such that $q_i \circ k = g_i$ for all i in $\mathcal{J}$. Now given a morphism $v : i \to j$ in $\mathcal{J}$, from the definition of g:

$$(p_v \circ g) = q_j$$

Then mulitplying by k:

$$(p_v \circ g) \circ k = q_j \circ k$$

But we know that $q_j \circ k = g_j$, hence:

$$(p_v \circ g) \circ k = g_j$$

Then as $< Y, g_i >$ is a cone, $g_j = Fv \circ g_i$, so:

$$(p_v \circ g) \circ k = Fv \circ g_i$$

We can now use the definition of k again, that is $q_i \circ k = g_i$:

$$(p_v \circ g) \circ k = Fv \circ q_i \circ k$$

But then from the definition of f, we now that $Fv \circ q_i = p_v \circ f$:

$$(p_v \circ g) \circ k = p_v \circ f \circ k$$

Then rearranging brackets, gives us:

$$p_v \circ (g \circ k) = p_v \circ (f \circ k)$$

Now using the universal property of the product $\prod_u Fcodu$, we know that:

$$f \circ k = g \circ k.$$

Therefore as e is an equalizer for f and g, there exists a unique morphism $h : Y \to X$, such that, $e \circ h = k$. We now need to show that $f_i \circ h = g_i$, for all i in $\mathcal{J}$, and that h is the unique morphism such that this is the case.

We know that $f_i = q_i \circ e$, so we want to show that:

$$q_i \circ e \circ h = g_i$$

But $e \circ h = k$, therefore we need:

$$q_i \circ k = g_i$$

But this follows from the definiton of k. So we see that h has the property that:

$$f_i \circ h = g_i$$

Now we just need to show that it is the unique such morphism. Suppose that we have another morphism, h' with the property that $f_i \circ h' = g_i$, for all i in $\mathcal{J}$, then much as above:

$$q_i \circ k = g_i = f_i \circ h' = (q_i \circ e) \circ h' = q_i \circ (e \circ h').$$

That is, $q_i \circ (e \circ h') = q_i \circ k$. But then the universal property of the product tells us that:

$$e \circ h' = k.$$

But since h was the unique morphism with this property, we can conclude that $h' = h$. Hence $< X, f_i >$ is a limit for this diagram.
□

Remark 0.3.57. The dual of the above proposition states that a category has all small co-limits iff it has all small co products and co-equalizers.

Definition 0.3.58. *A functor $F : \mathcal{C} \to \mathcal{D}$ is said to preserve limits of type $\mathcal{J}$ if whenever $p_j : L \to D_j$ is a limit for a diagram $D : \mathcal{C} \to \mathcal{C}$, the cone $Fp_j : FL \to FD_j$ is then a limit for the diagram $FD : \mathcal{J} \to \mathcal{C}$. A functor that preserves limits said to be 'continuous'.*

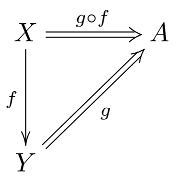
Definition 0.3.59. *A category is said to be small complete if all small limits exist. From the limit theorem, we can show that a category is complete by showing that it has all small products and equalizers. Similarly a category is complete if it has all limits.*

0.3.4 Yoneda lemma

Earlier on we defined the hom functors, these were functors, defined on a locally small category $\mathcal{C}$. There were two hom functors for a given object A in $\mathcal{C}$, one covariant and one contravariant. The first, the contravariant hom functor $Hom(-, A) : \mathcal{C} \to \mathbb{SET}$

$$Hom(-, A)(X) = Hom(X, A).$$

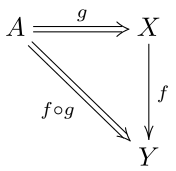
$$Hom(-, A)(f : X \to Y) = Hom(f, A) : Hom(Y, A) \to Hom(X, A).$$
$$Hom(f, A)(g : Y \to A) = g \circ f.$$



The second is the covariant hom functor $Hom(A,-) : \mathcal{C} \to \mathbb{SET}$

$$Hom(A,-)(X) = Hom(A,X).$$

$$Hom(A,-)(f : X \to Y) = Hom(A,f) : Hom(A,X) \to Hom(A,Y).$$
$$Hom(A,f)(g : A \to X) = f \circ g.$$



One possible question we can ask about the above definitions is whether these assignment are functorial, that is when we send an object A to $Hom(-,A)$, can we extend it to a functor? Let us first make it clear what such a functor would be like; as $Hom(-,A)$ is a contraviariant functor from $\mathcal{C} \to \mathbb{SET}$ it is an element of the functor category $[\mathcal{C}^{op}, \mathbb{SET}]$, therefore the functor we are looking for would be a functor $\mathcal{C} \to [\mathcal{C}^{op}, \mathbb{SET}]$.

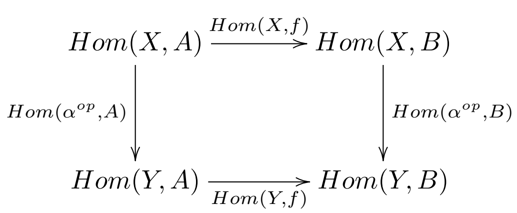
Let us provisionally call this functor $K : \mathcal{C} \to [\mathcal{C}^{op}, \mathbb{SET}]$. We already know how we want K to act on objects of $\mathcal{C}$, that is we want:

$$K(A) = Hom(-,A).$$

Therefore for a given morphism $f : A \to B$ in $\mathcal{C}$, we want Kf to be a morphism in $[\mathcal{C}^{op}, \mathbb{SET}]$, as $[\mathcal{C}^{op}, \mathbb{SET}]$ is a functor category and the morphisms in a functor category are natural transformations, we can conclude that Kf should be a natural transformation. That is in order that K be a functor it needs to send a morphism $f : A \to B$ in $\mathcal{C}$ to a natural transformation $Kf : Hom(-,A) \to Hom(-,B)$. Now we need to ask ourselves what such a natural transformation would be like. It would be a collection of morphisms in $\mathbb{SET}$, one for each object X in $\mathcal{C}$, that map:

$$Hom(X,A) \to Hom(X,B).$$

Fortuitously we already have just such a collection of functions, namely if we take the hom functor $Hom(X,-)$ (this is the covariant one, not the contrvariant one we were just using) and apply it to $f : A \to B$, then we get a function $Hom(X,A) \to Hom(X,B)$. Now it remains to show that this collection is a natural transformation. That is for all morphisms $\alpha^{op} : X \to Y$ in $\mathcal{C}^{op}$, the following diagram commutes:



Let us take $h : X \to A$, then the commutativity of this diagram comes down to requiring that:

$$Hom(\alpha^{op}, B) \circ Hom(X, f)(h) = Hom(Y, f) \circ Hom(\alpha^{op}, A)(h).$$

However we know that $Hom(X, f)(h) = f \circ h$, and $Hom(\alpha^{op}, A)(h) = h \circ \alpha$, hence this reduces to:

$$Hom(\alpha^{op}, B)(f \circ h) = Hom(Y, f)(h \circ \alpha).$$

We also know that $Hom(\alpha^{op}, B)(f \circ h) = (f \circ h) \circ \alpha$, and $Hom(Y, f)(h \circ \alpha) = f \circ (h \circ \alpha)$. So this reduces to the requirement that:

$$(f \circ h) \circ \alpha = f \circ (h \circ \alpha)$$

And then this follows trivially from the associativity requirement of the category $\mathcal{C}$.

So we see that Kf is a natural transformation, and all that remains to show that K is a functor is to show that it acts appropriately on idenitites and compositions. This K is called the Yoneda Embedding.

From now on we will denote the functor $Hom(-, A)$ by H_A, and the functor $Hom(A, -)$ by H^A.

Lemma 0.3.60. *Yoneda Lemma*
Suppose we have a category $\mathcal{C}$ which is locally small, then for any object C in $\mathcal{C}$ and functor $F : \mathcal{C}^{op} \to \mathbb{SET}$, there is an isomorphism:

$$\psi : Hom(H_C, F) \cong FC$$

Which is natural in C and F.

Proof. We first need to state how ψ acts, if we take $\theta : H_C \to F$, then we define:

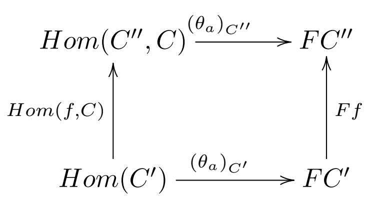
$$\psi_{C,F}(\theta) = \theta_C(id_C).$$

We denote this θ_x, that is $\theta_x = \theta_C(id_C)$. Now suppose that we are given some $a \in FC$, then we want to send this to a natural transformation $H_C \to F$, suppose that we are given an object C' in $\mathcal{C}$, then we define the natural transformation component wise:

$$(\theta_a)_{C'} : Hom(C', C) \to FC'.$$
$$(\theta_a)_{C'}(h) = F(h)(a).$$

For all $h : C' \to C$.

We need to check that this is a natural transformation, take $f : C'' \to C'$, then we need the following diagram to commute:



To see this, suppose we are given $h : C' \to C$ then:

$$(\theta_a)_{C''} \circ Hom(f, C)(h) = (\theta_a)_{C''}(h \circ f) = F(h \circ f)(a) = Ff \circ F(h)(a) = Ff(\theta_a)_{C'}h.$$

So we see that θ_a is a natural transformation, and the map θ well defined.

We now want to show that θ_x and x_θ are inverses. We begin with θ_{x_θ} for a given $\theta : H_C \to F$. Inserting defintions tells us that for any $h : C' \to C$:

$$(\theta_(x_\theta))_{C'}(h) = F(h)(\theta_C(id_C)).$$

But we know that θ is natural, hence the following commutes:

$$\begin{array}{ccc} H_C(C) & \xrightarrow{(\theta_C)} & FC \\ {\scriptstyle H_C(h)}\downarrow & & \downarrow{\scriptstyle Fh} \\ H_C(C') & \xrightarrow{(\theta_{C'})} & FC' \end{array}$$

Therefore:

$$(\theta_(x_\theta))_{C'}(h) = F(h)(\theta_C(id_C)) = \theta_{C'} \circ H_C(h)(id_C) = \theta_{C'}(h).$$

So we can conclude that $\theta_{x_\theta} = \theta$.

Suppose we are given some a in FC, then:

$$x_{\theta_a} = (\theta_a)_C(id_C) = F(id_C)(a) = id_{FC}(a) = a.$$

Hence we see that $Hom(H_C, F) \cong FC$.

We now need to show that this isomorphism has the required naturality properties. Suppose that we are given $\theta : F \to F'$, then we want the following diagram to commute:

$$\begin{array}{ccc} Hom(H_C, F) & \xrightarrow{\psi} & FC \\ {\scriptstyle Hom(H_C,\theta)}\downarrow & & \downarrow{\scriptstyle \theta_C} \\ Hom(H_C, F') & \xrightarrow{\psi} & F'C \end{array}$$

Taking ϕ in $Hom(H_C, f)$, then we need:

$$\theta_C(x_\phi) = \psi(Hom(H_C, \theta))(\phi)$$

But $x_\phi = \phi_C(id_C)$, hence:

$$\theta_C(x_\phi) = \theta_C(\phi_C(id_C)).$$

Then as $\theta_C(\phi_C) = (\theta \circ \phi)_C$:

$$\theta_C(x_\phi) = (\theta\phi)_C(id_C).$$

But we know that $x_{\theta\phi} = (\theta\phi)_C(id_C)$, so:

$$\theta_C(x_\phi) = x_{\theta\phi}.$$

Finally $\psi(Hom(H_C, \theta))(\phi) = x_{\theta\phi}$, hence:

$$\theta_C(x_\phi) = \psi(Hom(H_C, \theta))(\phi).$$

So we have establish naturality in F. For naturality in C, suppose that we are given $f : C' \to C$, then we require:

$$\begin{array}{ccc} Hom(H_C, F) & \xrightarrow{\psi} & FC \\ {\scriptstyle Hom(H_f,F)}\downarrow & & \downarrow{\scriptstyle Ff} \\ Hom(H_D, F) & \xrightarrow{\psi} & FD \end{array}$$

That is, for all $\theta : H_C \to F$:

$$\psi \circ Hom(H_f, F)(\theta) = Ff \circ \psi(\theta).$$

Putting in the definition of $Hom(H_f, F)$, means that we require:

$$\psi(\theta \circ H_f) = Ff \circ \psi(\theta).$$

Now, we assume the left hand side. From the definition of ψ:

$$\psi(\theta \circ H_f) = (\theta \circ H_f)_{C'}(id_{C'}).$$

Splitting up the right hand side:

$$\psi(\theta \circ H_f) = (\theta_{C'}) \circ (H_f)_{C'}(id_{C'}).$$

Then using this definition of H_f:

$$\psi(\theta \circ H_f) = (\theta_{C'})(f \circ id_{C'}) = (\theta_{C'})f = (\theta_{C'})(id_C \circ f).$$

Then $H_C(f)(id_C) = id_C \circ f$, giving us:

$$\psi(\theta \circ H_f) = (\theta_{C'})H_C(f)(id_C).$$

Now we know that θ is a natural transformation, therefore the following diagram commutes:

$$\begin{array}{ccc} H_C(C) & \xrightarrow{\theta_C} & FC \\ {\scriptstyle H_C(f)}\downarrow & & \downarrow{\scriptstyle Ff} \\ H_C(C') & \xrightarrow[\theta_{C'}]{} & F(C') \end{array}$$

Hence, in particular, $(\theta_{C'})H_C(f)(id_C) = Ff \circ \theta_C(id_C)$, meaning:

$$\psi(\theta \circ H_f) = Ff \circ \theta_C(id_C).$$

Finally using the definition of ψ:

$$\psi(\theta \circ H_f) = Ff \circ \psi(\theta).$$

Therefore we have established naturality in C. $\square$

Proposition 0.3.61. *The yoneda embedding $H_- : \mathcal{C} \to \mathbb{SET}^{\mathcal{C}^{op}}$ is full and faithful.*

Proof. For objects C, D in $\mathcal{C}$, we have an isomorphism:

$$Hom(C, D) = H_D C \cong Hom(H_C, H_D).$$

Moreover this isomorphism is induced by H_{-} it takes $h : C \to D$ to the natural transformation $\theta_h : H_C \to H_D$ given by:

$$(\theta_H)_{C'}(f : C' \to C) = H_D(f)(h) = Hom(f, D)(h) = h \circ f = (H_h)_{C'}(f).$$

Therefore $\theta_h = H_h$. $\square$

Corollary 0.3.62. *Suppose we have a locally small category $\mathcal{C}$, and objects A, B in $\mathcal{C}$, then:*

$$H_A \cong H_B \implies A \cong B.$$

Proof. This follows from the previous proposition, and the fact that for any full and faithful functor $F : \mathcal{C} \to \mathcal{D}$, $FA \cong FB \implies A \cong B$. $\square$

0.4 Adjoint Functors

0.4.1 Free Objects

In this section we present a collection of algebraic results relating to the construction of various free objects.
Our first example is the construction of the free monoid on a set.

Definition 0.4.1. *Given a set S, a word over S is a finite (possibly empty) string of elements of S. It will be denoted $[s_1, s_2, ..., s_n]$.*

$\triangleright$

Definition 0.4.2. *We then define the set S^*, to be the collection of all words over S, there are three important things to note about the set. Firstly, it contains the empty word, secondly all words are of finite length, the third is that within this set is a 'copy' of the original set, this is because all words of length one are just elements of the set on there own.*

$\triangleright$

Definition 0.4.3. *The free monoid on a set S, denoted $M(S)$, is the triple $(S^*, \cdot, -)$ where $\cdot$ denotes the concatenation of words, and $-$ denotes the empty word.*

$\triangleright$

For those not familiar with the term 'concatenation', it simply means joining two words together, suppose we have the word $[1, 2, 3]$ and the word $[4, 5]$, then $[1, 2, 3] \cdot [4, 5] = [1, 2, 3, 4, 5]$.

Proposition 0.4.4. *The free monoid is a monoid.*

Proof. First note that we do have a set, $M(S)$, and that $\cdot$ is a binary operation, that is it does take two words a produce another word that is also in the set. The empty word acts as the identity element, to see this, take any word $[s_1, ..., s_n]$ then if we add on the empty word $[\ ]$ we get:

$$[s_1, ..., s_n] \cdot [\ ] = [s_1, ..., s_n].$$
$$[\ ] \cdot [s_1, ..., s_n] = [s_1, ..., s_n].$$

Finally we need to show that $\cdot$ is associative, suppose that we have three words, $[s_1, ...s_k]$, $[t_1, ..., t_l]$, $[u_1, ..., u_m]$ all over the set S, then we need that:

$$([s_1, ...s_k] \cdot [t_1, ..., t_l]) \cdot [u_1, ..., u_m] = [s_1, ...s_k] \cdot ([t_1, ..., t_l] \cdot [u_1, ..., u_m]).$$

But, $([s_1, ...s_k] \cdot [t_1, ..., t_l]) \cdot [u_1, ..., u_m] = [s_1, ...s_k, t_1, ..., t_l] \cdot [u_1, ..., u_m] = [s_1, ...s_k, t_1, ..., t_l, u_1, ..., u_m]$.
And $[s_1, ...s_k] \cdot ([t_1, ..., t_l] \cdot [u_1, ..., u_m]) = [s_1, ...s_k] \cdot [t_1, ..., t_l, u_1, ..., u_m] = [s_1, ...s_k, t_1, ..., t_l, u_1, ..., u_m]$.
Hence :

$$([s_1, ...s_k] \cdot [t_1, ..., t_l]) \cdot [u_1, ..., u_m] = [s_1, ...s_k] \cdot ([t_1, ..., t_l] \cdot [u_1, ..., u_m]).$$

$\square$

The free monoid isn't just a nice way to construct a monoid from a given set, it has a very important universal property among monoid related to the set. Namely for any other monoid, N, and any set function, $g : S \to UN$ (where U is the forgetful functor on $\mathbb{MON}$), there exists a unique monoid homomorphism, $f : M(S) \to N$ such that:

$$g = U(f) \circ i.$$

(where i is the map $i : S \to UM(S)$, which sends s_1 to $U([s_1])$. We call this property the universal mapping property of the free monoid.

Proposition 0.4.5. *The free monoid, has the universal mapping property of the free monoid.*

Proof.
Suppose we have a monoid N, and a function $g : S \to UN$, then we first define a function $^* : UN \to N$, such that $(U(g))^* = g$, and then define a function $f : MS \to N$, by:

$$f(w) = \begin{cases} g(s_1)^* g(s_2)^* ... g(s_n)^* & \text{if } w = [s_1, s_2, ..., s_n] \\ (id_N) & \text{if } w \text{ is the empty word} \end{cases}$$

We need to show first that f is a homomorphism, that $g = U(f) \circ i$, and finally that f is the unique such homomorphism.

To see that f is a homomorphism, we need to check that it sends identities to identities and respects the operations of the monoids. Hence, as the identity of the free monoid is the empty word we need to show that f sends the empty word to the identity of N, but this follows from the definition. Secondly we need that for all w_1, w_2, for all words in MS:

$$f(w_1 w_2) = f(w_1) f(w_2)$$

We first consider the case when at least one of the words is empty, W.L.O.G. suppose that w_1 is empty, then clearly:

$$f(w_1 w_2) = f([] \cdot w_2) = f(w_2) = e_N f(w_2) = f(w_1) f(w_2).$$

Now assume that both of the words are non-empty, let:

$$w_1 = [s_1, ..., s_k], \text{ and } w_2 = [t_1, ..., t_l]$$

Then:

$$w_1 w_2 = [s_1, ..., s_k, t_1, ..., t_l].$$

And we have that,

$$f(w_1 w_2) = f([s_1, ..., s_k, t_1, ..., t_l]) = (g(s_1)...g(s_n))(g(t_1)...g(t_l)) = f([s_1, ..., s_k]) f([t_1, ..., t_l]) = f(w_1) f(w_2).$$

So we see that f is indeed a homomorphism.

Now we need to show that, $g = U(f) \circ i$, we can simply compute this:

$$(U(f) \circ i)(s) = (Uf(i(s)) = Uf([s]) = U(g(s)^*) = g(s).$$

Establishing the required identity.

So all that remains is to show that f is the unique such homomorphism. To this end, suppose that we have another homomorphism $h : MS \to N$, such that:

$$g = Uh \circ i_S.$$

Since we know that h is a homomorphism, it must be the case that $h([]) = id_N$, furthermore, we must have that $h(w_1w_2) = h(w_1)h(w_2)$. Now we also know that $g = Uh \circ i_S$, explicitly this means that for all s in S:

$$g(s) = (Uh \circ i_S)s.$$

However we know that:

$$i_S(s) = [s].$$

Hence:

$$g(s) = Uh([s]).$$

Then applying the map * to both sides gives us the identity:

$$g(s)^* = h([s])).$$

We now take any word $[s_1, ..., s_n]$ in MS, then:

$$f([s_1, ..., s_n]) = g(s_1)^* ... g(s_n)^* = h([s_1])...h([s_n]) = h([s_1, ..., s_n]).$$

Therefore we can conclude that $f = h$, and that f is the unique homomorphism such that $g = Uf \circ i_S$. $\square$

This mapping of sets to monoids actually makes up a functor, we call this functor the free functor on the category of monoids.

Definition 0.4.6. *The free functor on the category of monoids, $F_{\mathbb{MON}} : \mathbb{SET} \to \mathbb{MON}$, sends a set S to the free monoid, $M(S)$, and sends a function $f : S \to T$, to the monoid homomorphism defined by:*

$$Ff : M(S) \to M(T).$$
$$Ff([]) = [].$$
$$Ff([s_1, ..., s_n]) = [f(s_1), ..., f(s_n)].$$

$\triangleright$

Proposition 0.4.7. *The free functor is a functor.*

Proof. The above construction gave us a way to build a monoid $M(S)$, from any set S, so we see that the assignment $S \mapsto M(S)$ is well defined. The next thing we need to check is that for a given set function $f : S \to T$, the assigned $Ff : M(S) \to M(T)$ is a homomorphism, that is, $Ff(id_{M(S)}) = id_{M(T)}$, and $Ff(w_1w_2) = Ff(w_1)Ff(w_2)$.

The first one follows simply from the definition of Ff. For the second one, assume that we have two words $[a_1, ..., a_m]$ and $[b_1, ..., b_n]$ from the free monoid $M(S)$, then:

$$Ff([a_1, ..., a_m][b_1, ..., b_n]) = Ff([a_1, ..., a_m, b_1, ..., b_n]) = [f(a_1), ..., f(a_m), f(b_1), ..., f(b_n)] =$$
$$[f(a_1), ..., f(a_m)][f(b_1), ..., f(b_n)] = Ff([a_1, ..., a_m])Ff([b_1, ..., b_n]).$$

Therefore Ff is a homomorphism.

Now we need to check that for a given set S:

$$F(id_S) = id_{(M(S))}.$$

And given two functions $f : A \to B$, $g : B \to C$, then:

$$F(g \circ f) = Fg \circ Ff.$$

To see the first, note that $F(id_s)([s_1, ...s_n]) = [id(s_1), ..., id(s_n)] = [s_1, ..., s_n]$. To see the second, simply note that:

$$F(g \circ f)([s_1, ..., s_n]) = [g \circ f(s_1), ..., g \circ f(s_n)] = [g(f(s_1)), ..., g(f(s_n)] = Fg([f(s_1), ..., f(s_n)] = (Fg \circ Ff)[s_1, ..., s_n].$$

Therefore we conclude that the free functor is a functor.

□

Remark 0.4.8. Every integral domain can be turned into a field (the field of fractions). We will first work through the example of constructing the rationals from the integers. What do we mean by this? Surely we can just take the set $\{a/b \mid a, b \in \mathbb{Z} \ , \ b \neq 0\}$. However this doesn't quite work, the fractions have the property that $1/2 = 2/4$ even though $1 \neq 2$ and $2 \neq 4$, so we need a slightly more involved construction.

Definition 0.4.9. *The field of rationals $\mathbb{Q}$ is the field $\{\frac{a}{b} \mid a, b \in \mathbb{Z} \ , \ b \neq 0\}$ quotiented by the equivalence relation:*

$$\frac{a}{b} \sim \frac{c}{d} \text{ iff } ad = bc.$$

Where the operations are given by:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

The additive identity is the class $\frac{0}{b}$, the multiplicative identity is the class $\frac{a}{a}$, and the multiplicative inverse of an element $\frac{a}{b}$ is the element $\frac{b}{a}$.

▷

Proposition 0.4.10. *Addition is well defined*

Proof. Suppose that $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$, then we need that $\frac{ad + bc}{bd} \sim \frac{a'd' + b'c'}{b'd'}$

But, $\frac{a}{b} \sim \frac{a'}{b'} \implies ab' = a'b$ and $\frac{c}{d} \sim \frac{c'}{d'} \implies cd' = c'd$

Now,

$$\frac{ad + bc}{bd} \sim \frac{a'd' + b'c'}{b'd'}$$

iff

$$(ad + bc)(b'd') = (a'd' + b'c')(bd)$$

iff

$$(ad)(b'd') + (bc)(b'd') = (a'd')(bd) + (b'c')(bd)$$

iff

$$(ab')(dd') + (bb')(dd') + (bb')(cd') = (a'b)(dd') + (bb')(c'd)$$

iff

$$(a'b)(dd') + (bb')(c'd) = (a'b)(dd') + (bb')(c'd)$$

□

Proposition 0.4.11. *Multiplication is well defined*

Proof. Suppose $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$, we need that $\frac{ac}{bd} \sim \frac{a'c'}{b'd'}$.

$\frac{a}{b} \sim \frac{a'}{b'} \implies ab' = ba'$ and $\frac{c}{d} \sim \frac{c'}{d'} \implies cd' = dc'$.

$$\frac{ac}{bd} \sim \frac{a'c'}{b'd'}$$

iff

$$(ac)(b'd') = (a'c')(bd)$$

iff

$$(ab')(cd') = (a'c')(bd)$$

iff

$$(a'b)(c'd) = (a'c')(bd)$$

iff

$$(a'c')(bd) = (a'c')(bd)$$

□

Proposition 0.4.12. $\frac{0}{a}$, $a \neq 0$ *is the additive identity of* $\mathbb{Q}$

Proof. Let $\frac{b}{c}$ be an element of $\mathbb{Q}$, then $\frac{0}{a} + \frac{b}{c} = \frac{0c + ba}{ac} = \frac{ba}{ac} = \frac{b}{c}$.

□

Proposition 0.4.13. $\frac{a}{a}$, $a \neq 0$ *is the multiplicative identity of* $\mathbb{Q}$.

Proof. Let $\frac{b}{c}$ be an element of $\mathbb{Q}$, then $\frac{a}{a}\frac{b}{c} = \frac{ab}{ac} = \frac{b}{c}$

□

Proposition 0.4.14. *For an element* $\frac{a}{b}$ *of the field, where* a *is non-zero, the inverse is* $\frac{b}{a}$.

Proof. We simply compute:

$$\frac{a}{b}\frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = \frac{1}{1}$$

Which is the multiplicative identity. □

So we see that we have a field.

Definition 0.4.15. *For a given integral domain R the field of fractions is the field generated from the above construction, replacing $\mathbb{Z}$ with R.*

▷

Definition 0.4.16. *The localization of a integral domain. This functor, henceforth to be denoted $Q : \mathbb{INT} \to \mathbb{FIELD}$, sends an integral domain to it's field of fractions, and a morphism $f : I \to J$ is sent to the morphism $Qf : Quot(I) \to Quot(J)$, defined by $Qf : \frac{a}{b} \mapsto \frac{f(a)}{f(b)}$*

▷

Proposition 0.4.17. *The localization functor is a functor.*

Proof. Clearly Q sends objects in $\mathbb{INT}$ to objects in $\mathbb{FIELD}$. It remains to show that for a morphism $f : R \to S$ in $\mathbb{INT}$ Q sends f to a morphism $Qf : Q(R) \to Q(S)$ in $\mathbb{FIELD}$. we can see that Qf is definitely a function, so we need to show that it respects the field operations.

$$Qf\left(\frac{a}{b}+\frac{c}{d}\right) = Qf(\frac{ad+bc}{bd}) = \frac{f(ab+bd)}{f(bd)} = \frac{f(a)f(d)+f(b)f(c)}{f(b)f(d)} = \frac{f(a)}{f(b)} + \frac{f(c)}{f(d)} = Qf\left(\frac{a}{b}\right) + Qf\left(\frac{c}{d}\right)$$

$$Qf(\frac{a}{b})(\frac{c}{d}) = \frac{f(ac)}{f(bd)} = \frac{f(a)f(c)}{f(b)f(d)} = Qf\left(\frac{a}{b}\right)Qf\left(\frac{c}{d}\right)$$

So we see that Qf is a field morphism. We now need to check that the functor sends the identity to the identity, and respects composition.

$$Qid_R\left(\frac{a}{b}\right) = \frac{id_R(a)}{id_R(b)} = \frac{a}{b}.$$

$$Q(f\circ g)\left(\frac{a}{b}\right) = \frac{f\circ g(a)}{f\circ g(b)} = \frac{f(g(a))}{f(g(b))} = Qf\left(\frac{g(a)}{g(b)}\right) = Qf\left(Qg\left(\frac{a}{b}\right)\right) = Qf\circ Qg\left(\frac{a}{b}\right)$$ □

Remark 0.4.18. The field of fractions QD of an integral domain D has the following universal property, if $k : D \to F$ is an injective ring homomorphism from an integral domain D to a field F, then this extends to a a unique ring homomorphism $g : QD \to F$. That is given the map $i : D \to QD$, that sends $r \mapsto \frac{r}{1}$. For any field F, and injective ring homormophisms, $K : D \to UF$, there exists a unique field homomorphism $g : QD \to F$, such that:

$$k = Ug \circ i_D.$$

Where i_D is the inclusion associated with the domain D, and U is the forgetful functor.

0.4.2 Adjoint Functors

The examples that we have gone through above should strike you as similar, we will now formalize the process as that of adjunction.

Definition 0.4.19. *Given two categories, $\mathcal{C}$ and D, an adjunction between them is a pair of functors, $F : C \to D$ and $U : D \to C$ and a natural transformation, $\eta : id_C \to UF$ such that for all $g : x \to Uy$, $\exists! f : Fx \to y$ such that:*

$$g = Uf \circ \eta_x.$$

▷

We say that F is the left adjoint and U is the right adjoint. We write F ⊣ U. η is called the 'unit of adjuntion'.

Proposition 0.4.20. *The functor sending a $\mathbb{SET}$ to the free monoid on that $\mathbb{SET}$ is left adjoint to the forgetful functor $U : \mathbb{MON} \to \mathbb{SET}$.*

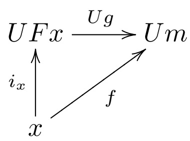
Proof. Let us first recap what we mean by a free monoid.
For any set X, we associated the monoid FX of all words over X, with concatenation of words as the binary operation, and the empty word acting as the identity element. There was also a function:

$$i_X : X \to UFX.$$

With the following UMP:

For any monoid, Y, and set function $f : X \to Uy$, there exists a unique homomorphism, $g : Fx \to y$ such that $f = Ug \circ i_x$. As described in the following diagram:

$$Fx \overset{g}{\dashrightarrow} y$$



So we see that $F : \mathbb{SET} \to \mathbb{MON}$ is left adjoint to $U : \mathbb{MON} \to \mathbb{SET}$ with unit $\eta : id_{\mathbb{SET}} \to UF$. □

Remark 0.4.21. If we take the category of all domains and injective ring homomorphisms, then the functor sending a domain to its field of fractions is left adjoint to the forgetful functor. $Q \dashv U$.

Lemma 0.4.22. *A function $F : A \to B$ is a bijection iff*

$$\forall b \in B, \exists!\ a \in A, \text{ such that, } f(a)=b.$$

Proof. The existence of such an a is equivalent to surjectivity, and the uniqueness is equivalent to injectivity. □

We now present another way to think of adjunctions. Suppose that we have an adjunction:

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

With unit $\eta : id_{\mathcal{C}} \to UF$, then if we are given two objects; X from $\mathcal{C}$ and Y from $\mathcal{D}$, we can form a map:

$$\phi_{(X,Y)} : Hom(FX, Y) \to Hom(X, UY).$$

By taking a morphism $f : FX \to Y$, and sending it to the morphism $Uf \circ \eta_c$. We know from the definition of adjunction and the previous lemma, that $\phi_{(X,Y)}$ is a bijection. (Or since we are working in the category $\mathbb{SET}$ we could say that this function is an isomorphism.)

We can denote this relation in the following diagram:

$$\frac{Fx \to y}{x \to Uy}$$

Proposition 0.4.23. *Given categories and functions:*

$$C \underset{U}{\overset{F}{\rightleftarrows}} D$$

then the following two conditions are equivalent.
1) F is left adjoint to U, $F \dashv U$.
2) For any object c in $\mathcal{C}$, and object d in $\mathcal{D}$, there is an isomorphism:

$$\phi : Hom(Fc, d) \cong Hom(c, Ud)$$

That is natural in both $\mathcal{C}$ and D.

Proof. We begin by showing that $1 \implies 2$, from the above discussion we already know that we have a bijection, so we just need to check that this bijection is natural in $\mathcal{C}$ and D. Take $h : c' \to c$, examine the following diagram:

$$\begin{array}{ccc} Hom_D(Fc, d) & \xrightarrow{\phi_{c,d}} & Hom_C(c, Ud) \\ \Big\downarrow{\scriptstyle Hom(Fh,d)} & & \Big\downarrow{\scriptstyle Hom(r,Ud)} \\ Hom_D(Fc', d) & \xrightarrow[\phi_{c',d}]{} & Hom_C(c', Ud) \end{array}$$

Then for $f : Fc \to D$, we have:

$$Hom(r, Ud) \circ \phi_{c,d} f = Hom(r, Ud) \circ (Uf \circ \eta_c) = (Uf \circ \eta_c) \circ h = Uf \circ (\eta_c \circ h) = Uf \circ (Ufh \circ \eta_{c'}) =$$
$$(Uf \circ Ufh) \circ \eta_{c'} = U(f \circ Fh) \circ \eta_{c'} = U(Hom(Fh, d)(f)) \circ \eta_{c'} = \phi_{c',d} \circ (Hom(Fh, d)(f))$$

This gives us naturality in $\mathcal{C}$, for naturality in $\mathcal{D}$, take $g : d \to d'$ and look at the following diagram:

$$\begin{array}{ccc} Hom_D(Fc, d) & \xrightarrow{\phi_{c,d}} & Hom_C(c, Ud) \\ \Big\downarrow{\scriptstyle Hom(Fc,g)} & & \Big\downarrow{\scriptstyle Hom(c,Ug)} \\ Hom_D(Fc, d') & \xrightarrow[\phi_{c,d'}]{} & Hom_C(c, Ud') \end{array}$$

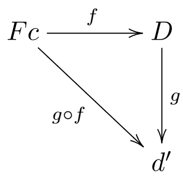
Now take $f : Fc \to d$, we have:

$$Hom(c, Ug)(\phi_{c,d}(f)) = Ug \circ (\phi_{c,d}(f)) = Ug \circ (U(f) \circ \eta_c) = (Ug \circ U(f)) \circ \eta_c = U(g \circ f) \circ \eta_c = U(Hom(Fc, g)(f)) \circ \eta_c = \phi_{c',d}(Hom(Fc, g)(f))$$

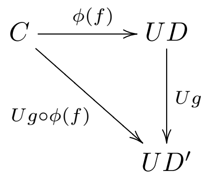
Hence 1 $\implies$ 2.
Now assume 2, this means that we have a bijection ϕ, such that:

$$\frac{Fx \to y}{x \to Uy}$$

Which is furthermore natural in $\mathcal{C}$ and $\mathcal{D}$, that is, for any commutative triangle :



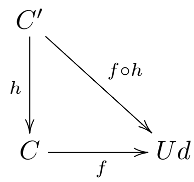
Then the two ways to get an arrow, $c \to Ud'$ are the same, i.e. we can take $\phi(g \circ f)$, or we can form the triangle:



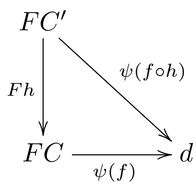
So naturality in $\mathcal{D}$ means that:

$$\phi(g \circ f) = Ug \circ \phi(f)$$

Dually, naturality in $\mathcal{C}$ means that given:



The following commutes:



That is,

$$\psi(f \circ h) = \psi(f) \circ F(h)$$

Now we want to construct a natural transformation

$$\eta : 1_C \to U \circ F.$$

With the required universal property. To this end, we know that we have a bijection between morphisms $Fc \to d$ and morphisms $c \to Ud$, but as Fc is just an object of $\mathcal{D}$, there must be a bijection between morphisms $Fc \to Fc$ and morphisms $c \to UFc$.

$$\frac{Fc \to Fc}{c \to UFc}$$

But we know that there must always be an identity morphism $id_{Fc} : Fc \to Fc$, hence we can define η_c to be $\phi(id_{Fc})$.

We still need to check that this fulfills the universal property of the unit, that is for all $g : Fc \to d$, as we are assuming that ϕ is a bijection, we need that :

$$\phi(g) = U(g) \circ \eta_c$$

But:

$$Ug \circ \eta_c = Ug \circ \phi(id_{Fc}).$$

Then from the triangle above:

$$Ug \circ \eta_c = \phi(g \circ id_{Fc}) = \phi(g).$$

□

Remark 0.4.24. The first definition has a dual. That is we can give a third characterization of a adjuntion. Given categories:

$$C \underset{U}{\overset{F}{\rightleftarrows}} D$$

$F \dashv U$ iff there is a natural transforamtion:

$$\epsilon : FU \to id_{\mathcal{D}}$$

Such that:

For any c in $\mathcal{C}$, d in $\mathcal{D}$, and $g : Fc \to d$, there exists a unique morphism $f : c \to Ud$, where $g = \epsilon_d \circ Ff$.

In which case ϵ is said to be the counit of the adjuntion, and we have the identity:

$$\epsilon_d = \psi(id_{Ud})$$

Where $\psi = \phi^{-1}$.

We now give the new 'official' defintion of adjoint functors.

Definition 0.4.25. *An adjunction consists of functors and categories:*

$$C \underset{U}{\overset{F}{\rightleftarrows}} D$$

and a natural isomorphism:

$$\phi : Hom(Fc, d) \to Hom(c, Ud) : \psi$$

The unit and counit are then:

$$\eta_c = \phi(1_{Fc})$$
$$\epsilon_d = \psi(1_{Ud})$$

▷

Example 0.4.26. The diagonal functor, $\Delta : \mathcal{C} \to \mathbf{1}$, sends each object $c \in \mathcal{C}$ to the element $1 \in \mathbf{1}$. We can ask what it would mean for a functor to be right adjoint to the diagonal functor. It would have to be a functor $U : \mathbf{1} \to \mathcal{C}$, which moreover establishes the bijection:

$$\frac{\Delta c \to 1}{c \to U1}$$

However $\Delta c = 1$, so a morphism $\Delta c \to 1$ has to be the identity morphism $id_1 : 1 \to 1$. Therefore what we have is that for every $c \in \mathcal{C}$ there exists a bijection between morphisms $id_1 : 1 \to 1$, and morphisms $c \to U1$, but for each $c \in \mathcal{C}$ there exists a unique morphism $1 \to 1$,
i.e. for each $c \in \mathcal{C}$, there exists a unique morphism $c \to U1$ in $\mathcal{C}$, hence $U1$ is a terminal object in $\mathcal{C}$.
So what we have established is that for a given category $\mathcal{C}$, there exists a right adjoint to the diagonal functor $\Delta_1 : \mathcal{C} \to \mathbf{1}$, iff $\mathcal{C}$ has a terminal object.

◇

Given that terminal and initial objects are dual we should expect that we can also characterize initial objects in terms of adjoints, however it might not be obvious how we should do this, perhaps we should consider the diagonal functor that acts into the empty category. It turns out however that we should look at the left adjoint for the same diagonal functor.

Example 0.4.27. A left adjoint to the diagonal functor $\Delta_1 : \mathcal{C} \to \mathbf{1}$ is a functor $F : \mathbf{1} \to \mathcal{C}$ that sends 1 to an initial object in $\mathcal{C}$.
To see this note that a left adjoint to this functor is a functor $F : \mathbf{1} \to \mathcal{C}$, and a bijection:

$$\frac{F1 \to c}{1 \to \Delta c}$$

However Δc is 1, hence for each $c \in \mathcal{C}$, we have is a bijection between morphisms $1 \to 1$ and morphisms $F1 \to c$, but as there exists a unique morphisms $1 \to 1$, we end up with is that for each $c \in \mathcal{C}$, there exists a unique morphism $F1 \to c$, therefore $F1$ is an initial object of $\mathcal{C}$.

◇

Example 0.4.28. We now look at the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, which sends each $A \in \mathcal{C}$ to $< A, A >$ in $\mathcal{C} \times \mathcal{C}$, and each morphism $f : A \to B$ to the following morphisms in $\mathcal{C} \times \mathcal{C}$:

$$< f, f >: A \times A \to B \times B.$$

We claim that the product functor is right adjoint to this functor. The product functor is the functor:

$$\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

Which sends a pair of objects A, B, to the product $A \times B$, and a pair of morphisms $f : A \to B$, $g : C \to D$, to the product morphism $< f, g >: A \times C \to B \times D$.

In order to show that the product functor is right adjoint we need to establish the following bijection:

$$\frac{< A, A > \to < B, C >}{A \to B \times C}$$

That is a bijective mapping of pairs of morphisms, $f : A \to B$, $g : A \to C$, to a morphism $\phi(f, g) : A \to B \times C$, that is natural in $\mathcal{C}$ and $\mathcal{C} \times \mathcal{C}$.

We know that a single morphism $h :< A, A > \to < B, C >$ is actually an ordered pair of morphisms in $\mathcal{C}$:

$$h_1 : A \to B,\ h_2 : A \to C.$$

At this point we can use the fact that $B \times C$ is a product to establish the existence of a morphism $k : A \to B \times C$, this is the unique morphism such that:

$$p_1 \circ k = h_1, \text{ and } p_2 \circ k = h_2.$$

We then use this property to define ϕ, that is for a given $k :< A, A > \to < B, C >$,

$$\phi(k) = h.$$

Where h is the unique morphism such that $p_1 \circ k = h_1$, and $p_2 \circ k = h_2$.

From the product property we know that this is a bijection, so in order to conclude that the product functor is right adjoint, we simply need to check the naturality conditions, that is, for all morphisms, $f : A \to B$ in $\mathcal{C}$, and $< g_1 : C \to E, g_2 : D \to F >$ in $\mathcal{C} \times \mathcal{C}$, the following diagrams commute:

$$\begin{array}{ccc}
Hom(< A, A >, < Y, Z >) & \xrightarrow{\phi_{(A,<Y,Z>)}} & Hom(A, Y \times Z) \\
\uparrow {\scriptstyle Hom(<f,f>,<Y,Z>)} & & \uparrow {\scriptstyle Hom(f,<Y,Z>)} \\
Hom(< B, B >, < Y, Z >) & \xrightarrow[\phi_{(B,<Y,Z>)}]{} & Hom(B, Y \times Z)
\end{array}$$

$$\begin{array}{ccc}
Hom(< A, A >, < C, D >) & \xrightarrow{\phi_{(A,<C,D>)}} & Hom(A, C \times D) \\
\downarrow {\scriptstyle Hom(<A,A>,<g_1,g_2>)} & & \downarrow {\scriptstyle Hom(A,<g_1,g_2>)} \\
Hom(< A, A >, < E, F >) & \xrightarrow[\phi_{(A,<E,F>)}]{} & Hom(A, E \times F)
\end{array}$$

The first diagram requires that for all morphisms pairs of morphisms $k_1 : B \to Y$, $k_2 : B \to Z$, the following identity holds:

$$Hom(f, < Y, Z >) \circ \phi_{(B,<Y,Z>)} < k_1, k_2 >= \phi_{(A,<Y,Z>)} \circ Hom(< f, f >, < Y, Z >) < k_1, k_2 >$$

Then inserting the definiton of $Hom(< f, f >, < Y, Z >)$ and $Hom(f, < Y, Z >)$ this becomes:

$$\phi_{(B,<Y,Z>)}(< k_1, k_2 >) \circ f = \phi_{(A,<Y,Z>)}(< k_1 \circ f, k_2 \circ f >)$$

We know that $\phi_{(B,<Y,Z>)}(< k_1, k_2 >) =< k_1, k_2 >$, and that $\phi_{(A,<Y,Z>)}(< k_1 \circ f, k_2 \circ f >) =< k_1 \circ f, k_2 \circ f >$, where the first $< k_1 \circ f, k_2 \circ f >$ is an ordered pair and the second is the unique morphism specified by the product. Putting these into the above we see that we need:

$$< k_1, k_2 > \circ f =< k_1 \circ f, k_2 \circ f >$$

We know that $< k_1 \circ f, k_2 \circ f >$ is the unique morphisms mapping $A \to Y \times Z$ such that:

$$p_1 \circ < k_1 \circ f, k_2 \circ f >= k_1 \circ f$$
$$p_2 \circ < k_1 \circ f, k_2 \circ f >= k_2 \circ f$$

But,

$p_1 \circ (< k_1, k_2 > \circ f) = (p_1 \circ < k_1, k_2 >) \circ f = k_1 \circ f$, $p_2 \circ (< k_1, k_2 > \circ f) = (p_2 \circ < k_1, k_2 >) \circ f = k_2 \circ f$.

Hence we conclude that $< k_1, k_2 > \circ f =< k_1 \circ f, k_2 \circ f >$, and we see that the first diagram commutes.

The second diagram commutes if, for all pairs of morphisms, $l_1 : A \to C$, $l_2 : A \to D$, the following identity holds:

$$Hom(A, < g_1, g_2 >) \circ \phi_{(A,<C,D>)} < l_1, l_2 >= \phi_{(A,<E,F>)} \circ Hom(< A, A >, < g_1, g_2 >) < l_1, l_2 >$$

Then putting in the defintion of $Hom(< A, A >, < g_1, g_2 >)$ and $Hom(A, < g_1, g_2 >)$, we get:

$$< g_1, g_2 >) \circ \phi_{(A,<C,D>)}(< l_1, l_2 >) = \phi_{(A,<E,F>)} \circ (< g_1, g_2 > \circ < l_1, l_2 >)$$

We then note that $< g_1, g_2 > \circ < l_1, l_2 >=< g_1 \circ l_1, g_2 \circ l_2 >$ giving us:

$$< g_1, g_2 >) \circ \phi_{(A,<C,D>)}(< l_1, l_2 >) = \phi_{(A,<E,F>)} \circ (< g_1 \circ l_1, g_2 \circ l_2 >)$$

Now we can use the definition of ϕ, to get the requirement that:

$$< g_1, g_2 > \circ < l_1, l_2 >=< g_1 \circ l_1, g_2 \circ l_2 >$$

but this simply reduces to the tautology:

$$< g_1 \circ l_1, g_2 \circ l_2 >=< g_1 \circ l_1, g_2 \circ l_2 >$$

Therefore we conclude that $\Delta \dashv \times$.

$\diamond$

Definition 0.4.29. *For a small index category $\mathcal{J}$, and a category $\mathcal{C}$, the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$, is the functor that sends an object A of $\mathcal{C}$ to the functor $\Delta(A)$ in $\mathcal{C}^{\mathcal{J}}$, where:*

$$\Delta(A)(j) = A, \textit{ for all objects } j \textit{ in } \mathcal{J}.$$
$$\Delta(A)(u) = id_A, \textit{ for all morphisms } u \textit{ in } \mathcal{J}.$$

$\triangleright$

Proposition 0.4.30. $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ *is a functor.*

Proof. To see this we need to show that, for all objects A in $\mathcal{C}$, $\Delta(A)$ is a functor, and moreover that this assignment is functorial. To see that $\Delta(A)$ is a functor $\mathcal{J} \to \mathcal{C}$, first note that it sends objects of $\mathcal{C}$ to objects of $\mathcal{C}$, and that it sends a morphism $u : i \to j$, to a morphism:

$$\Delta(A)(u) : \Delta(A)(i) \to \Delta(A)(j).$$

This is because $\Delta(A)(u) = id_A$, $\Delta(A)(i)$, $\Delta(A)(j)$, and hence the requirement actually says:

$$id_A : A \to A.$$

Which we know to be true. So all that remains in order to verfiy that $\Delta(A)$ is a functor is to check that it sends identities to idenitties, and that it respects composition:

$$\Delta(A)(id_i) = id_A$$
$$\Delta(A)(u \circ v) = id_A = id_A \circ id_A = \Delta(A)(u) \circ \Delta(A)(v).$$

So we see that $\Delta(A)$ is indeed a functor.

We now see that Δ sends objects of $\mathcal{C}$ to objects of $\mathcal{C}^{\mathcal{J}}$. But we still need to define how Δ acts on morphisms of $\mathcal{C}$, that is we need to define $\Delta(f)$ for a morphism f in $\mathcal{C}$ so that $\Delta(f)$ is a morphism in $\mathcal{C}^{\mathcal{J}}$. We already know that the morphisms of a functor category are natural transformations. So we see that we need to take a morphism $f : A \to B$ in $\mathcal{C}$ and send it to a natural transformation:

$$\Delta(f) : \Delta(A) \to \Delta(B).$$

What would such a natural transformation look like? It would be a collection of morphisms in $\mathcal{C}$, $\Delta(f)_i : \Delta(A)(i) \to \Delta(B)(i)$, one for each object i in $\mathcal{J}$. But from the definition of $\Delta(A)i$ and $\Delta(B)i$, this simplifies to:

$$\Delta(f)_i : A \to B$$

So it seems like we can define each $\Delta(f)_i$ to just be f. We now need to check that this satisfies the commutativity properties. That is for all morphisms $u : i \to j$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{CD}
\Delta(A)(i) @>{\Delta(f)_i}>> \Delta(B)(i) \\
@V{\Delta(A)(u)}VV @VV{\Delta(B)(u)}V \\
\Delta(A)(j) @>>{\Delta(f)_j}> \Delta(B)(j)
\end{CD}$$

But we know that, $\Delta(A)(i) = A$, $\Delta(A)(j) = A$, $\Delta(B)(i) = B$, $\Delta(B)(j) = B$, $\Delta(A)(u) = id_A$, $\Delta(B)(u) = id_B$, $\Delta(f)_i = f$ and $\Delta(f)_j = f$, so we see that we actually need the following diagram to commute:

$$\begin{CD}
A @>{f}>> B \\
@V{id_A}VV @VV{id_B}V \\
A @>>{f}> B
\end{CD}$$

That is, $id_B \circ f = f \circ id_A$, which clearly holds. Hence, $\Delta(f)$ is a natural transformation, and the above assignment is correct. So all that remains is to check that Δ sends identities to identities and that it respects composition.
So we need that:

$$\Delta(id_A) = id_{\Delta(A)}$$
$$\Delta(f \circ g) = \Delta(f) \circ \Delta(g).$$

But as $\Delta(id_A) = \{id_A\}$ and $id_{\Delta(A)} = \{id_A\}$, we see that these two are the same. So we just need to check that it respects composition.
We know that:

$$\Delta(f \circ g) = \{f \circ g\}$$
$$\Delta(f) = \{f\},\ \Delta(g) = \{g\}.$$

Hence $\Delta(f) \circ \Delta(g) = \{f \circ g\} = \Delta(f \circ g)$. Therefore Δ is a functor.
□

We now define a new functor. Suppose that we are working in a category $\mathcal{C}$, and that we are given some index category $\mathcal{J}$. Then a diagram for $\mathcal{C}$, is a functor $D : \mathcal{J} \to \mathcal{C}$, but this is precisely an object of $\mathcal{C}^{\mathcal{J}}$. So we see that if the category $\mathcal{C}$ has all limits of type $\mathcal{J}$, then this gives us a way to assign objects of $\mathcal{C}^{\mathcal{J}}$ to objects of $\mathcal{C}$, that is we would send a diagram to its limit.

At this point we would like to extend this function to a functor $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. So we send a natural transformation $\eta : D_1 \to D_2$ of $\mathcal{C}^{\mathcal{J}}$ (i.e. a morphism of this category) to a morphism in $\mathcal{C}$, that maps:

$$lim(\eta) : limF \to limG.$$

If we take an object i of $\mathcal{J}$, then we can form the following diagram:

$$\begin{array}{c} limF \\ \Big\downarrow {\scriptstyle f_i} \\ Fi \\ \Big\downarrow {\scriptstyle \eta_i} \\ Gi \\ \Big\uparrow {\scriptstyle g_i} \\ limG \end{array}$$

Therefore if we take the cone $limf = (limf, f_i)$ then we can create a new cone, $limf' = (limf, \eta_i \circ f_i)$. As this cone maps to the diagram G, and we know that $limG$ is a limit for this diagram, we can conclude that there exists a unique morphism $k : limF \to limG$, such that:

$$g_i \circ k = \eta_i \circ f_i.$$

We now say that the functor lim sends a natural transformation $\eta : F \to G$, to this unique morphism k.

Proposition 0.4.31. *For category $\mathcal{C}$, and a small index category $\mathcal{J}$, the functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ is left adjoint to the functor $lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. That is:*

$$\Delta \dashv lim.$$

Proof. We say that $\Delta \dashv lim$ when we have a natural isomorphism:

$$\frac{\Delta(C) \to D}{C \to limD}$$

Before we begin to show that there is such a natural isomorphism, it would be helpful to pause and briefly recap what exactly such a natural isomorpshim would look like. First note that a morphism $\Delta(C) \to D$ is a natural transformation between the functors $\Delta(C)$ and D, where $\Delta, D : \mathcal{J} \to \mathcal{C}$. So we see that both of these are actually diagrams of type $\mathcal{J}$ in the category $\mathcal{C}$.

A morphism $C \to limD$ is simply a morphism in $\mathcal{C}$. What the isomorphism ϕ tells us is that there is a bijection between natural transformations $\Delta(C) \to D$ and morphisms $C \to limD$, that is in some sense natural.

A natural transformation $\eta : \Delta(C) \to D$ is a collection of morphisms $\eta_i : \Delta(C)(i) \to D(i)$, in $\mathcal{C}$, one for each object i in $\mathcal{J}$. However $\Delta(C)(i) = C$, for all i in $\mathcal{J}$. Hence we actually have a collection of morphisms:

$$\eta_i : C \to D(i)$$

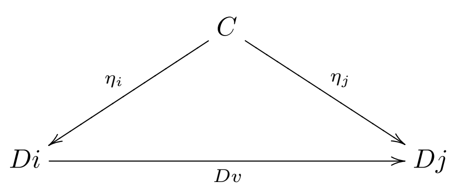
One for each object i in $\mathcal{J}$. But we know that D is a diagram, therefore what we really have is a cone (We haven't shown that it commutes in an appropriate way with morphisms Dv, but we won't worry about that now). That is to say, a natural transformation $\Delta(C) \to D$ is a cone $< C, \eta_i >$ for the diagram D.

Hence what this hypothetical natural isomorphism gives us is a bijection between cones over an object C, and morphisms from C to $limD$, that is in some way natural.

With the above as our guide, we will go ahead and define a map ϕ which sends a cone $< C, \eta_i >$ over a diagram D, to the morphism $h : C \to limD$, which is the unique morphism $C \to limD$, such that $p_i \circ h = \eta_i$, for all i in $\mathcal{J}$. Now, we know from the definition of a limit for a diagram that this is a well defined map, and that moreover, from the uniqueness property of this morphism, that this map is a bijection. What we now want to do is show that this is a map:

$$Hom(\Delta(C), D) \to Hom(C, lim(D)).$$

But because of the work that we have done above, in motivating the definition, we just need to show that any natural transformation $\Delta(C) \to D$ is indeed a cone in the technical sense. We saw that a natural transformation $\eta : \Delta(C) \to D$, gave us an object C in $\mathcal{C}$, and a collection of morphisms in $\mathcal{C}$, $\eta_i : C \to Di$, one for each i in $\mathcal{J}$, so we just need to show that for any morphism $v : i \to j$ in $\mathcal{J}$, the following commutes:



There is only really one thing we can attempt at this point, and that is to use the fact that η is a natural transformation. We know that for any morphism $v : i \to j$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccc} \Delta(C)i & \xrightarrow{\eta_i} & Di \\ {\scriptstyle \Delta(C)v}\downarrow & & \downarrow{\scriptstyle Dv} \\ \Delta(C)j & \xrightarrow[\eta_j]{} & Dj \end{array}$$

Then into this commutative diagram we can put the fact that $\Delta(C)i = C$, $\Delta(C)j = C$, to obtain the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\eta_i} & Di \\ {\scriptstyle id_C}\downarrow & & \downarrow{\scriptstyle Dv} \\ C & \xrightarrow[\eta_j]{} & Dj \end{array}$$

Which is precisely what we required (all be it with a slightly different orientation). So we see that an element of $Hom(\Delta(C), D)$ is a cone over the diagram D.

To recap what we have shown so far, we have established that a natural isomorphism $\phi : Hom(\Delta(C), D) \to Hom(C, limD)$, is a bijection of cones over D and morphisms fom the object of the cone to $limD$, we then showed that we could use the property of a limit, that for any cone there exists a unique commuting morphism, to define a bijection $C \to limD$. However we have still not yet shown that the functors Δ, and lim are adjoint. We need to show that this isomorphism is a natural isomorphism.

That is for every morphism $f : A \to B$ in $\mathcal{C}$, and $g : X \to Y$, in $\mathcal{C}^{\mathcal{J}}$, the following diagrams commute:

$$\begin{array}{ccc} Hom(\Delta(B), X) & \xrightarrow{\phi_{B,X}} & Hom(B, limX) \\ {\scriptstyle Hom(\Delta f, X)}\downarrow & & \downarrow{\scriptstyle Hom(f, limX)} \\ Hom(\Delta(A), X) & \xrightarrow[\phi_{A,Y}]{} & Hom(A, limX) \end{array}$$

$$\begin{array}{ccc} Hom(\Delta(A), X) & \xrightarrow{\phi_{A,X}} & Hom(A, limX) \\ {\scriptstyle Hom(\Delta(A), g)}\downarrow & & \downarrow{\scriptstyle Hom(A, limg)} \\ Hom(\Delta(A), Y) & \xrightarrow[\phi_{A,Y}]{} & Hom(A, limY) \end{array}$$

The first says that for an natural transformation $\eta : \Delta(B) \to X$:

$$Hom(f, limX) \circ \phi_{B,X}(\eta) = \phi_{A,Y} \circ Hom(\Delta f, X)(\eta).$$

Inserting the definition of the contravariant hom functors give us:

$$\phi_{B,X}(\eta) \circ f = \phi_{A,Y}(\eta \circ \Delta f).$$

The '$\eta \circ \Delta f$' is a composition of natural transformations, hence $(\eta \circ \Delta f)_i = \eta_i \circ (\Delta f)_i$. Then we can use $(\Delta f)_i = f$ for all i, so that $\eta \circ \Delta f$ is a cone $< A, \eta_i \circ f >$. (We clearly have a collection of morphisms in $\mathcal{C}$, mapping A to Di, one for each i in $\mathcal{J}$, commutativity comes from $Dv \circ \eta_i = \eta_j \implies Dv \circ \eta_i \circ f = \eta_j \circ f \implies Dv \circ (\eta \circ \Delta)_i = (\eta \circ \Delta f)_j$). So we know that $\phi_{A,Y} \circ Hom(\Delta f, X)(\eta)$ will be the unique morphism $h : A \to limY$ such that for all i in $\mathcal{J}$:

$$p_i \circ h = (\eta_i \circ f)$$

So we see that in order to establish the commuataivity of the diagram, it would suffice to show that $\phi_{B,X}(\eta) \circ f$ (which we can see maps $A \to limY$) also satisfies this identity for all i in $\mathcal{J}$. By definition $\phi_{B,X}(\eta)$ is the unique morphism $k : B \to limY$, such that for all i in $\mathcal{J}$:

$$p_i \circ k = \eta_i$$

But then multiplying on the right by f gives us:

$$p_i \circ k \circ f = \eta_i \circ f$$

For all i in $\mathcal{J}$, therefore, $k \circ f = h$. Yet as $h = \phi_{A,Y}(\eta \circ \Delta f)$, and $k = \phi_{B,X}(\eta)$ this is exactly what we wanted. Hence we have verified the commutativity of the first diagram.

We now want to establish the commuativity of the second diagram, recall that we had a morphism $g : X \to Y$, and we wanted for all $\tau : \Delta(A) \to X$):

$$Hom(A, limg) \circ \phi_{A,X}(\tau) = \phi A, Y \circ Hom(\Delta(A), g)(\tau)$$

Putting in the definiton of the covariant hom functors tells us that we require:

$$limg \circ \phi_{A,X}(\tau) = \phi_{A,Y} \circ (g \circ \tau)$$

We know that g is a morphism in $^{\mathcal{J}}$, hence it is a natural transformation, so $g \circ \tau : \Delta(A) \to Y$ is a cone $< A, (g \circ \tau)_i >$ for the diagram Y (it is a collection of morphisms $(g \circ \tau)_i = g_i \circ \tau_i : A \to Yi$, to see commuativity as a cone follows from commutativity as a natural transformation). Therefore $\phi_{A,Y} \circ (g \circ \tau)$ is the unique morphism $h : A \to limY$, such that, for all i:

$$p_i \circ h = (g \circ \tau)_i.$$

(p_i is the projection from $limY$ to Yi) Now, $\phi_{A,X}(\tau)$ is the unique morphism $k : A \to limX$, such that for all i:

$$q_i \circ k = \tau_i.$$

(q_i is the projection from $limX$ to Xi) So we wish to show that $p_i \circ (limg \circ k) = (g \circ \tau)_i$, at which point we could conclude that $limg \circ k = h$.
Recall that given a natural transformation $g : X \to Y$, $limg$ is the unique morphism $limg : limX \to limY$, such that:

$$p_i \circ limg = g_i \circ q_i.$$

The multiplying on the right by k tells us that we have:

$$p_i \circ limg \circ k = g_i \circ q_i \circ k.$$

But we know that $q_i \circ k = \tau_i$, hence:

$$p_i \circ (limg \circ k) = g_i \circ \tau_i$$

However $g_i \circ \tau_i = (g \circ \tau)_i$, therefore:

$$p_i \circ (limg \circ k) = (g \circ \tau)_i$$

Which is what we wanted. Hence $(limg \circ k) = (g \circ \tau)_i$, and we know that the second diagam commutes. Therefore we have established that the isomorphism ϕ is natural, and that $\Delta \dashv lim$. □

Example 0.4.32. We now look at a special case of an adjoint, where the categories are also partial orders.
Suppose we have two posets, P and Q, then a Galois connection between them, consists of two monotone functions $F : P \to Q$, $U : Q \to P$, such that $\forall a \in A$, $b \in B$:

$$Fa \leq b \text{ iff } a \leq Ub.$$

Hence we see that if we consider these two posets as categories, then the monotone maps are nothing more than functors, and the condition on them simply tells us that they are adjoint functors.

◇

0.4.3 Adjoint Functor Theorem

Proposition 0.4.33. *Adjoints are unique up to isomorphism, that is given a functor $F : \mathcal{C} \to \mathcal{D}$, and two right adjoints $U, V : \mathcal{D} \to \mathcal{C}$:*

$$F \dashv U \text{ and } F \dashv V.$$

Then $U \cong V$.

Proof. For any D in $\mathcal{D}$, C in $\mathcal{C}$, since $F \dashv U$:

$$Hom_{\mathcal{C}}(C, UD) \cong Hom_{\mathcal{D}}(FC, D).$$

Then since $F \dashv V$

$$Hom_{\mathcal{D}}(FC, D) \cong Hom_{\mathcal{C}}(C, VD).$$

Hence from the yoneda lemma, $UD \cong VD$. □

This means that we can actually characterise an object or construction by requiring that it be left or right adjoint to a given functor. For example, we know that the product functor is left adjoint to the diagonal functor, hence all left adjoints to the diagonal functor are isomorphic to the product functor.

Proposition 0.4.34. *Right Adjoints preserve limits, left adjoints preserve colimits.*

Proof. Suppose we have an adjunction:

$$C \underset{U}{\overset{F}{\rightleftarrows}} D \qquad\qquad F \dashv U$$

and a diagram $D : \mathcal{J} \to \mathcal{D}$ such that $lim_{\leftarrow} D_j$ exists in $\mathcal{D}$. Then for any X in $\mathcal{C}$, as $F \dashv U$:

$$Hom_{\mathcal{C}}(X, U(lim_{\leftarrow} D_j)) \cong Hom(FX, lim_{\leftarrow} D_j).$$

Then as Hom preserves limits:

$$Hom_{\mathcal{C}}(X, U(lim_{\leftarrow} D_j)) \cong lim_{\leftarrow} Hom_{\mathcal{D}}(FX, D_j).$$

But as $F \dashv U$, we have $lim_{\leftarrow} Hom_{\mathcal{C}}(X, UD_j) \cong lim_{\leftarrow} Hom_{\mathcal{D}}(FX, D_j)$:

$$Hom_{\mathcal{C}}(X, U(lim_{\leftarrow} D_j)) \cong lim_{\leftarrow} Hom_{\mathcal{C}}(X, UD_j).$$

Then as Hom preserves limits:

$$Hom_{\mathcal{C}}(X, U(lim_{\leftarrow} D_j)) \cong Hom_{\mathcal{C}}(X, lim_{\leftarrow} UD_j).$$

Therefore from Yoneda's lemma, $U(lim_{\leftarrow} D_j) \cong lim_{\leftarrow} UD_j$.

□

Definition 0.4.35. *In the following lemma we are going to need a generalization of the notion of the equalizer of two morphisms. Suppose we are given the category $\mathcal{J}$, with two objects A, B, and a collection of morphisms:*

$$(\alpha_i : A \to B)_{i \in I}$$

Indexed by the set I. Suppose we are also given a diagram $D : \mathcal{J} \to \mathcal{C}$, of type $\mathcal{J}$, a cone for this diagram would be a triple:

$$< X, f_A, f_B >.$$

Where $f_A : X \to DA$, $f_B : X \to DB$, and where $\alpha_i \circ f_A =_B$ for all i in I.

So we see that what the cone actually gives us is an object X of $\mathcal{C}$, and a morphism $f_A : X \to A$, such that $\alpha_i \circ f_A = \alpha_j \circ f_A$ for all i and j in I. That is we have the joint equalizer of the morphisms α_i.

▷

Lemma 0.4.36. *Let $\mathcal{D}$ be locally small and complete, then the following two conditions are equivalent:*
- *$\mathcal{D}$ has an initial object.*
- *$\mathcal{D}$ satisfies the following 'solution set condition:*

There is a set of object $(D_i)_{i \in I}$ in $\mathcal{D}$, such that for any object D in $\mathcal{D}$, there is an arrow $D_i \to D$, for some D_i in I.

Proof. We begin by assuming that $\mathcal{D}$ has an initial object 0, then we can take the set $\{0\}$, clearly for any object D in $\mathcal{D}$, there exists a morphism $0 \to D$ and hence the solution set criterion is satisfied.

Now suppose that the solution set criterion is satisfied. That is we have a collection of objects $(D_i)_{i \in I}$ such that for any object D in $\mathcal{D}$, there exists a morphism $D_i \to D$, for some i in I. As $\mathcal{D}$ is complete, we can take the product $W = \prod_{i \in I} D_i$. Then this object has the property that for any object D in $\mathcal{D}$, there exists a morphism (not neccessarily unique) that maps:

$$W \to D.$$

This is because the projection morphisms give us a morphism $W \to D_i$, and then the solution set criterion gives us a morphism $D_i \to D$, the composition of these two giving us a morphism $W \to D$.

We now form the product $\prod_{d: W \to W} W$ over all morphisms $d \to d$, that is we form a product made up of $|\{d : W \to W\}|$ lots of W. We can from a cone for this product by taking an endomorphism $d' : W \to W$, and considering the collection $< W, d' >$, the product property then gives is a unique morphism $< d' >: W \to \prod_{d: W \to W} W$ such that for all $d : W \to W$:

$$p_d \circ < d' >= d'$$

We mark out one particular morphism of this type, that is if we take the cone formed from the endomorphism $id_W : W \to W$, then we get a unique morphism which we will call $\Delta : W \to \prod_{d:W\to W} W$. Such that for all $d : W \to W$:

$$p_d \circ < d' >= id_w.$$

We now take the joint equalizer $h : V \to W$, of all the morphisms $< d >: W \to \prod_{d:W\to W} W$. Then from the definition of the equalizer, we must have that:

$$< d > \circ h = \Delta \circ h.$$

Mulitplying on the left by some p_d tells us that:

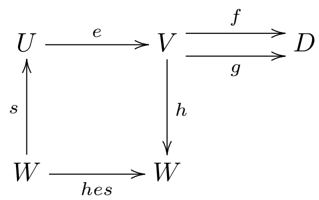
$$p_d \circ < d > \circ h = p_d \circ \Delta \circ h.$$

But we know that $p_d \circ < d >= d$, and $p_d \circ \Delta = id_W$, so we get the identity that for all endomorphisms $d : W \to W$:

$$d \circ h = h.$$

V also 'weakly initial', that is, for every object D in $\mathcal{D}$, there exists a morphism from V to every other object D. This follows from the fact that we know that W is weakly initial and we know that we have a morphism from V to W. We now wish to show that this morphism is unique, and hence that V is an initial object in this category.

Suppose that there are two morphisms $f, g : V \to D$, then we can take the equalizer $e : U \to V$ of these two morphisms. As we said above, we know that W is weakly initial, hence if we select a morphism $s : W \to U$, we get the following diagram:



But then from the fact that $d \circ h = h$, and that hes is an endomorphism $W \to W$, we must have that:

$$h(esh) = h.$$

But we know that all equalizers are monic, and that h is an equalizer, hence:

$$esh = id_V.$$

Then we can mulitply on the right by e:

$$eshe = e.$$

Yet e is also monic, hence:

$$she = id_U.$$

Therefore $U \cong V$, since e was an equalizer $U \to V$, and $f = g$. Hence V is an initial object. $\square$

Theorem 0.4.37. *Freyd's adjoint functor theorem.*
Let $\mathcal{C}$ be locally small, and complete, then given any category $\mathcal{X}$, and functor U such that:

$$U : \mathcal{C} \to \mathcal{X}$$

Then the following are equivalent:
- *U has a left adjoint.*
- *U preserves limits, and for each $X \in \mathcal{X}$, the functor U satisfies the following solution set criterion:*

There exists a set of objects $(C_i)_{i \in I}$ in $\mathcal{C}$, such that for all C in $\mathcal{C}$, and $f : X \to UC$, there exists $i \in I$, $\phi : X \to UC_i$, $f' : C_i \to C$, such that :
$$f = U(f') \circ \phi$$

Proof. We begin by supposing that $F \dashv U$. Then $\{FX\}$ is a solution set for X. To see we need to check that for all C in $\mathcal{C}$ and $f : X \to UC$, there exists FX, $\phi : X \to UFX$, $f' : FX \to C$ such that:

$$f = U(f') \circ \phi.$$

But we can take ϕ to be the unit of adjunction η which for a given X maps $X \to UFX$. But this is then follows from the definition of adjunction.

For the opposite implication, we consider the comma category, $< X \downarrow U >$. Then, U has a left adjoint if the comma category $< X \downarrow U >$ has an initial object. (this because the initial object $< FX, \eta : A \to UFX >$ has the ump of the unit).

Now we can use the previous proposition to show that $< X \downarrow U >$ has an initial object, that is we need to show that:
1) $< X \downarrow U >$ is locally small.
2) $< X \downarrow U >$ satisfies the solution set criterion for this lemma.
3) $< X \downarrow U >$ is complete.
Condition 1 follows from the fact that $\mathcal{C}$ is locally small. From the solution set criterion from this theorem there is a set of objects:

$$\{(C_i, \phi : X \to UC_i) \; | i \in I\}.$$

Such that every object $(C, f : X \to UC)$ has an arrow $f' :< C_i, \phi > \to < C, f >$.
Finally to see that $< X \downarrow U >$ is complete, we know that $\mathcal{C}$ has all small products and equalizers, furthermore we know that U preserves limits, hence from the limit theorem, $< X \downarrow U >$ is complete.
□

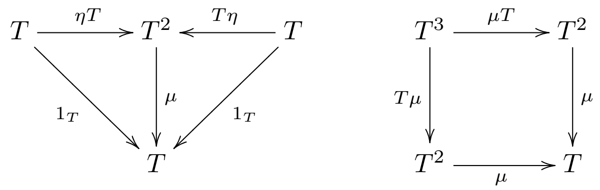
0.5 Monads

0.5.1 Monads

Definition 0.5.1. *A monad on a category $\mathcal{C}$ is a triple, $< T, \eta, \mu >$ where T is a functor $T : \mathcal{C} \to \mathcal{C}$, and η, μ, are natural transformations $\eta : 1_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ satisfying the two conditions:*

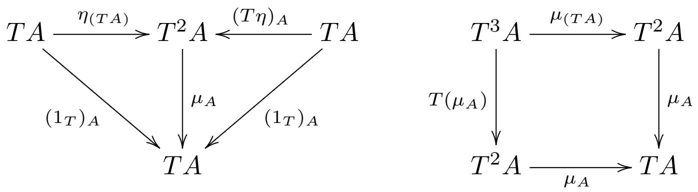
$$\mu \circ \eta T = 1 = \mu \circ T\eta \text{ - Unit Law}$$
$$\mu \circ \eta T = \mu \circ T\eta \text{ - Associativity Law}$$

i.e.



▷

Remark 0.5.2. The conditions that we have given in the definition of a monad require certain natural transformations to be equal, what we want on the level of morphisms is that for all objects A in $\mathcal{C}$:



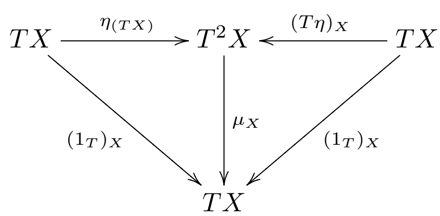
Example 0.5.3. The monad of monoids. This is the triple $< T, \eta, \mu >$, where T is the functor $T : \mathbb{SET} \to \mathbb{SET}$ which sends a set X, to the set X^*, made up of all words over X. η is the natural transformation that for a set X, sends an element of the set to the word of length one containing the element, and μ is the natural transformation which for a set X, concatenates two words.

Remark 0.5.4. To give you an idea of what X^* is like, if we take the set $X = \{0, 1\}$, then:

$$X^* = \{[], [0], [1], [0,0], [0,1], [1,1], [0,0,0], [0,0,1]...\}.$$

$$\eta_X(0) = [0].$$
$$\eta_X(1) = [1].$$
$$\mu([[x_1, x_2, ..., x_k], [x_1', x_2', ..., x_l'], ..., [x_l'', x_1'', ..., x_m'']]) = [x_1, x_2, ..., x_k, x_1', x_2', ..., x_1'', ...x_m''].$$

We need to show that this triple satisfies the associativity and unit laws. The unit law states that, for every set X, $\mu_X \circ \eta T_X = 1 = \mu_X \circ T\eta_X$:



That is for all $[x_1, ..., x_n]$ in TX.:

$$(\mu_X \circ \eta_{TX})([x_1, ..., x_n]) = (i_T)_X([x_1, ..., x_n]).$$
$$(\mu_X \circ (T\eta)_X)([x_1, ..., x_n]) = (i_T)_X([x_1, ..., x_n]).$$

However we know that, $(i_T)_X([x_1, ..., x_n]) = [x_1, ..., x_n]$, so what we want is:

$$(\mu_X \circ \eta_{TX})([x_1, ..., x_n]) = [x_1, ..., x_n].$$
$$(\mu_X \circ (T\eta)_X)([x_1, ..., x_n]) = [x_1, ..., x_n].$$

The natural transformation $\eta_X : X \to TX$ takes an element x and sends it to $[x]$ in X^*, hence the natural transformation $(\eta T)_X : TX \to TX^*$, acts by taking an element a in TX and sends it to $[a]$ in TX^*. However $TX = T^*$, and $TX^* = X^{**}$. Therefore $(\eta T)_X : X^* \to X^{**}$ sends an element $[x_1, ..., x_n]$ to $[[x_1, ..., x_n]]$.

We now need to look at how μ_X acts on $[[x_1, ..., x_n]]$, recall that μ_X takes a list of lists, and joins up the lists inside; so it must be that:

$$\mu([[x_1, ..., x_n]]) = [x_1, ..., x_n].$$

If you can't see this then imagine taking $[[x_1, ...x_n]]$ and replacing it with the 'list of lists', $[[x_1, ..., x_n], [], [], ..., []]$, then clearly this list is the same as the original, and if we apply μ_X to it, we end up with $[x_1, ..., x_n]$. Hence we see that:

$$(\mu_X \circ \eta_{TX})([x_1, ..., x_n]) = [x_1, ..., x_n].$$

As $\eta_{TX}([x_1, ..., x_n]) = [[x_1, ..., x_n]]$, and $\mu([[x_1, ..., x_n]]) = [x_1, ..., x_n]$.

Now we need to check the second identity. That states that:

$$(\mu_X \circ (T\eta)_X)([x_1, ..., x_n]) = [x_1, ..., x_n].$$

We first need to make clear in what way the functor $T : \mathbb{SET} \to \mathbb{SET}$ acts on functions. If given a set function $f : A \to B$ between two sets A and B then we define the function Tf to be the set function $Tf : A^* \to B^*$:

$$Tf[a_1, ..., a_n] = [f(a_1), ...f(a_n)].$$

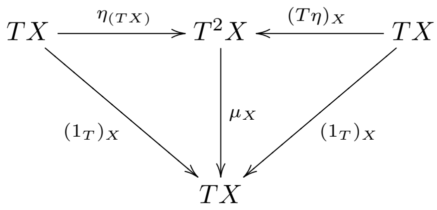
Once again, the natural transformation $\eta_X : X \to TX$ takes an element x and sends it to $[x]$ in X^*, the natural transformation $T\eta_X : TX \to TX^*$, is defined to be the map: $T(\eta_X)$, that is the image of η_X under T, hence from how we defined T above:

$$(T\eta)_X([x_1, ..., x_n]) = [\eta(x_1), ..., (\eta(x_n)] = [[x_1], ..., [x_n]].$$

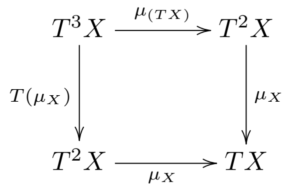
And clearly if we apply μ to $[[x_1], ..., [x_n]]$ we end up with $[x_1, ..., x_n]$. Hence we have established the second identity:

$$(\mu_X \circ (T\eta)_X)([x_1, ..., x_n]) = [x_1, ..., x_n].$$

So the work that we have done so far has established the unit law:



It still remains to show that this monad satisfies the associativity law. The associativity law states that for every set X:



$$\mu_X \circ \mu T_X = \mu_X \circ T\mu_X$$

These are functions $T^3X \to TX$, that is, functions mapping 'words of words of words ' to 'words'.

So we want that for every element of T^3X,
$Z = [\ [[a_1, ..., a_k], ..., [z_1, ..., z_m]], ..., [[(a_1)', ..., (a_k)'], ..., [(z_1)', ..., (z_m)']]\ ]$:

$$(\mu_X \circ \mu T_X)(Z) = (\mu_X \circ T\mu_X)(Z).$$

We know that $Tf(A, B, ..., K]) = [f(A), f(B), ..., f(K)]$, hence $T\mu_X(Z)$ becomes:

$$[\ \mu_X([[a_1, ..., a_k], ..., [z_1, ..., z_m]]), ..., \mu_X([[(a_1)', ..., (a_k)']), ..., \mu_X([(z_1)', ..., (z_m)']])\ ].$$

Which upon accounting for the action of μ_X becomes:

$$[\ [a_1, ..., a_k, ..., z_1, ..., z_m], ..., [(a_1)', ..., (a_k)', ..., (z_1)', ..., (z_m)']\ ].$$

And then applying μ_X to this we get:

$$[a_1, ..., a_k, ..., z_1, ..., z_m, ..., (a_1)', ..., (a_k)', ..., (z_1)', ..., (z_m)']:$$

So satisfying the associativity law comes down to showing that the right hand side of the above equation is equal to this value. that is that:

$$(\mu_X \circ \mu T_X)(Z) = [a_1, ..., a_k, ..., z_1, ..., z_m, ..., (a_1)', ..., (a_k)', ..., (z_1)', ..., (z_m)'].$$

We know that $(\mu T_X)(Z) = (\mu)_{TX}(Z)$, hence:

$$(\mu)_{TX}(Z) = [[a_1, ..., a_k], ..., [z_1, ..., z_m], ..., [(a_1)', ..., (a_k)'], ..., [(z_1)', ..., (z_m)']]$$

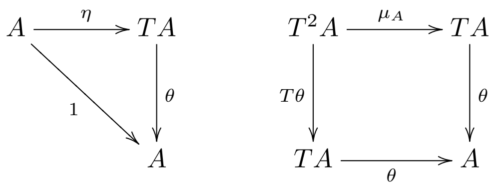
And then applying μ_X to this gives us:

$$[a_1, ..., a_k, ..., z_1, ..., z_m, ..., (a_1)', ..., (a_k)', ..., (z_1)', ..., (z_m)']$$

Hence the associativity identity holds, and we have verified that this is indeed a monad.

◇

Definition 0.5.5. *An algebra for a monad* $< T, \eta, \mu >$ *is a pair* $< A, \theta >$ *where* A *is an object of* $\mathcal{C}$ *and* θ *is a morphism* $\theta : TA \to A$ *of* $\mathcal{C}$*. Such that the following diagrams commute:*



▷

Example 0.5.6. Taking the monad of monoids defined above, an algebra for this monad is then an object of the category $\mathbb{SET}$ (a set A) and a morphism of $\mathbb{SET}$, $\theta : TA \to A$, i.e. a function θ mapping $A^* \to A$. Furthermore, this pair needs to satisfy the two axioms, that is:

$$(\theta \circ \eta)(a) = a, \text{ for all } a \in A$$
$$(\theta \circ \mu_A)(b) = (\theta \circ T\theta)(b), \text{ for all } b \in T^2A.$$

But recall that $\eta(a) = [a]$, hence the first requirement is that:

$$\theta([a]) = a, \text{ for all } a \in A.$$

And recalling that μ concatenated a collection of words into one word, means that for a given $b = [[x_1, ..., x_n], ..., [z_1, ..., z_m]]$ in T^2A:

$$(\theta \circ \mu_A)([[x_1, ..., x_n], ..., [z_1, ..., z_m]]) = (\theta \circ T\theta)([[x_1, ..., x_n], ..., [z_1, ..., z_m]]).$$

But $\mu_A([[x_1, ..., x_n], ..., [z_1, ..., z_m]]) = [x_1, ..., x_n, ..., z_1, ..., z_m]$, hence we need:

$$\theta([x_1, ..., x_n, ..., z_1, ..., z_m]) = (\theta \circ T\theta)([[x_1, ..., x_n], ..., [z_1, ..., z_m]]).$$

And $T\theta([[x_1, ..., x_n], ..., [z_1, ..., z_m]]) = [\theta([x_1, ..., x_n]), ..., \theta([z_1, ..., z_m])]$, hence:

$$\theta([x_1, ..., x_n, ..., z_1, ..., z_m]) = \theta([\theta([x_1, ..., x_n]), ..., \theta([z_1, ..., z_m])]).$$

The set A, along with the binary operation given by $a \cdot b = \theta([a, b])$, is a monoid.

We clearly have a set, and as θ maps $A^* \to A$, this is a valid binary operation, so we need to check that there is an identity element, and that the operation is assocciative.
We get the associativity requirement of the monoid, by using the second requirement on an algebra, for a given a_1, a_2, a_3, we can form the word, $[[a_1, a_2], [a_3]]$, then:

$$\theta[a_1, a_2, a_3]) = \theta([\theta([a_1, a_2]), \theta([a_3])].$$

That is:

$$a_1 \cdot a_2 \cdot a_3 = (a_1 \cdot a_2) \cdot a_3$$

We now repeat the above process with the word $[[a_1], [a_2, a_3]]$, giving us the identity:

$$a_1 \cdot a_2 \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$$

And then combining these allows us to conclude that:

$$(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$$

So we see that A fulfills the associativity requirement. Finally we need to check that A has an identity. We know that there is an empty word in the set TA, and we know that θ must map this empty word to some element in A. We want to show that the image of $[]$ under θ acts as a unit, to this end, consider the word $[[], [a]]$ in T^2A, once again using the second requirement:

$$(\theta \circ \mu_A)([[], [a]]) = (\theta \circ T\theta)([[], [a]]).$$

Then, as $\mu_A([[], [a]]) = [a]$, and $T\theta([[], [a]]) = [\theta([]), \theta([a])]$.

$$\theta([a]) = \theta[\theta([]), \theta([a])].$$

We also know that $\theta([a]) = a$, hence:

$$a = \theta[\theta([]), a].$$

Putting in the definition of θ:

$$a = \theta([]) \cdot a.$$

Therefore $\theta([])$ is a left identity, to see that $\theta([])$ is a right identity, consider the word, $[[a], []]$, which tells us that:

$$a \cdot \theta([]) = a.$$

Hence, we see that an algebra for this monad is a monoid.

◇

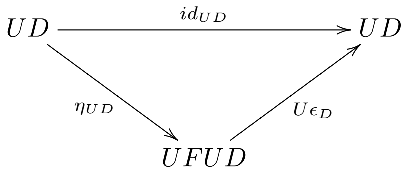
We are now going to provide a third way to charactierise adjoint functors. Suppose we have an adjunction $F \dashv U$, for functors:

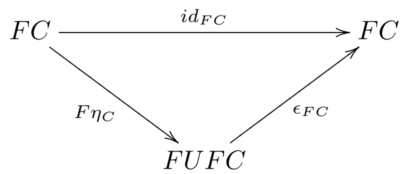
$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

With unit and counit,

$$\eta : id_{\mathcal{C}} \to UF$$
$$\epsilon : FU \to id_{\mathcal{D}}.$$

Then we know that $\phi^{-1}(\epsilon_D) = id_{UD}$, hence, $id_{UD} = \phi(\epsilon_D) = U(\epsilon_D) \circ \eta_{UD}$. Similarly, $id_{FC} = \phi^{-1}(\eta_C) = \epsilon_{FC} \circ F(\eta_C)$. Giving us the following two 'triangle identities'.





We now prove that these conditions are actually sufficient for a adjunction.

Proposition 0.5.7. *Suppose we have the following:*

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$$
$$\eta : id_{\mathcal{C}} \to UF$$
$$\epsilon : FU \to id_{\mathcal{D}}$$

Then if ithe triangle identities hold, $F \dashv U$.

Proof. In order to show that $F \dashv U$, we need a natural isomorphism:

$$\phi : Hom_{\mathcal{D}}(FC, D) \to Hom_{\mathcal{C}}(C, UD)$$

Then we use the definition:

$$\phi(f : FC \to D) = U(f) \circ \eta_C$$
$$\psi(g : C \to UD) = \epsilon_D \circ Fg.$$

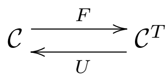
We can see that these two functions are inverse,

$$\phi(\psi(g)) = \phi(\epsilon_D \circ Fg) = U(\epsilon_D) \circ UFf \circ F\eta_C = U(\epsilon_D) \circ \eta_{UD} \circ g = g.$$
$$\psi(\phi(f)) = \psi(Uf \circ \eta_C) = \epsilon_D \circ FU(f) \circ F\eta_C = f \circ \epsilon_{FC} \circ F\eta_C = f$$

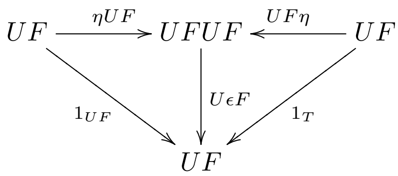
Furthermore, this isomorphism is natural. $\square$

Proposition 0.5.8. *Every adjunction gives rise to a monad.*

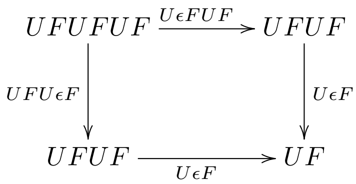
Proof. Suppose we have adjoint functors:



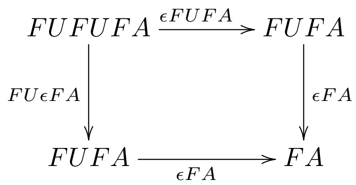
Then take $< UF, \eta, U\epsilon F >$ we need to check that these fulfill the two axioms:



But the right one is just U of a triangle identity. and the one on the left is just the other triangle identity with F on the right.
The second condition is:



To see this, ϵ is a natural transformation $\epsilon : UF \to id_{\mathcal{C}}$, therefore for the morphism ϵFA:



Then applying U to this:

$$\begin{array}{ccc} UFUFUFA & \xrightarrow{U\epsilon FUFA} & UFUFA \\ \Big\downarrow{\scriptstyle UFU\epsilon FA} & & \Big\downarrow{\scriptstyle U\epsilon FA} \\ UFUFA & \xrightarrow[U\epsilon FA]{} & UFA \end{array}$$

And we see that this holds for all objects A in $\mathcal{C}$, therefore we have satisfied the associativity requirement. $\square$

We saw above that every adjunction gives rise to a monad, we might also ask, given a monad can we construct an adjunction? It turns out that there are two important ways that this can be done. Given any monad T, we have the category of algebras for it.

Definition 0.5.9. *For a monad $< T, \eta, \mu >$, given two algebras, $D : TA \to A$, $E : TB \to B$, a morphism $f' : D \to E$, consists of, a morphism $f : A \to B$, such that the following commutes:*

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle E} \\ A & \xrightarrow[f]{} & B \end{array}$$

$\triangleright$

Lemma 0.5.10. *Suppose that we have three algebras, $D : TA \to A$, $E : TB \to B$, and $F : TC \to C$. If we are given two morphisms of algebras, $g' : E \to F$, and $f' : D \to E$, then their composition $g' \circ f'$ (construed by composing the underlying morphisms of $\mathcal{C}$) is a morphism of algebras, $g' \circ f' : D \to F$.*

Proof. As $g : B \to C$, and $f : A \to B$, their composition, $g \circ f : A \to C$, hence all that we need to check is that:

$$\begin{array}{ccc} TA & \xrightarrow{T(g \circ f)} & TC \\ \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle F} \\ A & \xrightarrow[g \circ f]{} & C \end{array}$$

But we know that the following diagrams commute:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle E} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} TB & \xrightarrow{Tg} & TC \\ \Big\downarrow{\scriptstyle E} & & \Big\downarrow{\scriptstyle F} \\ B & \xrightarrow[g]{} & C \end{array}$$

Therefore:

$$F \circ T(g \circ f) = F \circ Tg \circ Tf = g \circ E \circ Tf = g \circ f \circ D = (g \circ f) \circ D.$$

Therefore, the composition of two 'morphisms of algebras' is a 'morphism of algebras'. $\square$

Proposition 0.5.11. *The category $\mathcal{C}^T$ of all algebras over T, and morphisms of algebras is a category.*

Proof. We have a well defined collection of objects, and also a well defined collection of morphisms. Moreover each morphisms has a domain and codomain. The above lemma tells us that composition is well defined, so all that remains is to show the existence of an identity morphism for a given algebra, and to show that composition of morphisms is associative.

We will first show that composition of morphisms is associative. Suppose that we have three composable morphisms h', g' and f', then we want to show that:

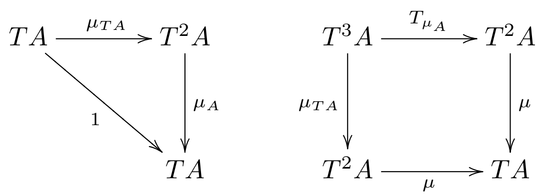
$$(h' \circ g') \circ f' = h' \circ (g' \circ f').$$

But this follows from the fact that composition of mophisms is defined by composing the underlying morphisms of $\mathcal{C}$, which is known to be associative.

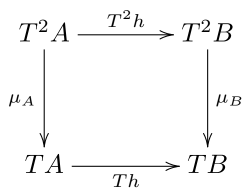
Finally we need to show that there is an identity morphism for every algebra, but given an algebra $D : TA \to A$, ${id_A}' : D \to D$ is an identity, as when we compose this morphism with any other morphism of algebras f', it equates to composing f with id_A.
□

Now that we have formed a category, we want to establish an adjunction using this category. We define the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$, to be the functor that sends an object $< A, \theta >$ of $\mathcal{C}^T$ to A, and sends a morphism of algebras $f' :< A, \theta_1 > \to < B, \theta_2 >$ to the underlying morphism $f : A \to B$. Now, we are going to show that this forgetful functor is right adjoint to the free functor.

Definition 0.5.12. *For a given monad T, we define the free functor $F : \mathcal{C} \to \mathcal{C}^T$, which sends an object A in $\mathcal{C}$, to the free algebra $< TA, \mu_A >$. We need to check that $< TA, \mu_A >$ is a T-algebra. We know that TA is an object of $\mathcal{C}$, and that μ_A is a morphism of $\mathcal{C}$, mapping, $T^2A \to TA$. So we just need to check that the two following diagrams commute:*



But these follow directly from the definition of a monad. We now need to specify how F acts on morphisms of $\mathcal{C}$. If we are given $h : A \to B$ in $\mathcal{C}$, then as μ is a natural transformation:



But this tells us that Th is a T-algebra homomorphism $< TA, \mu_A > \to < TB, \mu_B >$. The we say that $Fh = Th$. So the functor F sends objects to objects and morphism to morphisms in the required way. We just need to see that it acts apprpriately on identities and compositions.

$$F(id_A) = T(id_A) = id_TA$$
$$F(g \circ f) = T(g \circ f) = T(g) \circ T(f).$$

So we see that F is a functor.

▷

Proposition 0.5.13. *If we are given:*

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C}^T$$

Then $F \dashv U$.

Proof. We need to find a natural transformations $\eta' : id_{\mathcal{C}} \to UF$ that acts as the unit of adjunction. But;

$$UFC = U(< TC, \mu_C >) = TC.$$

Hence, we need a natural transformation $\eta' : id_{\mathcal{C}} \to T$. So hopefully we can use the natural transformation $\eta : id_{\mathcal{C}} \to T$ that we already have. We also want a counit, that is a natural transformation $\epsilon : FU \to id_{\mathcal{C}^T}$. Once again, $FU(< C, \theta >) = FC =< TC, \mu_C >$, hence we want a map of algebras, $\epsilon_{<C,\theta>} :< TC, \mu_C >\to< C, \theta >$ the only thing that we can really try is the morphism $\theta : TC \to C$.

So we need to check first that ϵ is a natural transformations, and then we need to check that η and ϵ fulfill the triangle identities.

To see that ϵ is a natural transformation first note that for all objects $< A, \theta >$ of $\mathcal{C}^T$, ϵ gives us a morphism of $\mathcal{C}^T$, mapping $TA \to A$ as required. We also need that for all morphisms of algebras $g' :< A, \theta_2 >\to< B, \theta_3 >$ the following diagram commutes:

$$\begin{array}{ccc} < TA, \mu_A > & \xrightarrow{\epsilon_{<A,\theta_2>}} & < A, \theta_2 > \\ {\scriptstyle Tg}\downarrow & & \downarrow{\scriptstyle g'} \\ < TB, \mu_B > & \xrightarrow[\epsilon_{<B,\theta_3>}]{} & < B, \theta_3 > \end{array}.$$

On the level of morphisms in $\mathcal{C}$, this equates to:

$$\begin{array}{ccc} TA & \xrightarrow{\theta_2} & A \\ {\scriptstyle Tg}\downarrow & & \downarrow{\scriptstyle g} \\ TB & \xrightarrow[\theta_3]{} & B \end{array}.$$

But, as we know that $g' :< A, \theta_2 >\to< B, \theta_3 >$ is a morphism of algebras, the commutativity of the above diagram follows simply from the commutativity condition for a morphism of algebras. Hence we have verified that μ and ϵ are natural transformations with the required domains and codomains. So in order to establish that $F \dashv U$, we simply need to verify the triangle identities. That is:

$$id_F = \epsilon F \circ F\eta$$
$$id_U = U\epsilon \circ \eta U.$$

Which means that, for all objects C in $\mathcal{C}$, and $< A, \theta >$ in $\mathcal{C}^T$ we require:

$$id_{FC} = \epsilon_{FC} \circ F\eta_C.$$
$$id_{U(<A,\theta>)} = U(\epsilon_{<A,\theta>}) \circ \eta_{(U<A,\theta>)}$$

Beginning with the first identity, $\epsilon_{FC} = \mu_C$ hence:

$$id_{FC} = \mu_C \circ F\eta_C.$$

And we know how F acts on morphisms, that is, $F\eta_C = T\eta_C$, therefore:

$$id_{FC} = \mu_C \circ T\eta_C.$$

But this falls out of the unit law for monads.

We now need to check the second identity, that is:

$$id_{U(<A,\theta>)} = U(\epsilon_{<A,\theta>}) \circ \eta_{(U<A,\theta>)}$$

But, we know that $U(< A, \theta >) = A$, hence we actually require:

$$id_A = U(\epsilon_{<A,\theta>}) \circ \eta_{(U<A,\theta>)}$$

We also know that $\epsilon_{<A,\theta>} = \theta$ and that $\eta_{(U<A,\theta>)} = \eta_A$, therefore we need:

$$id_A = U(\theta) \circ \eta_A$$

But we know that this is the case, form the conditions imposed on θ by its being a morphism of algebras. Hence we can conclude that $F \dashv U$. $\square$

There is another important way in which we can construct an adjunction from a monad, namely, through the use of the Kleisli category. Suppose that we have a monad $< T, \eta, \mu >$ over a category $\mathcal{C}$, then we form the Kleisli category $\mathcal{C}_T$, which is comprised of all the objects of $\mathcal{C}$, and the morphisms between two objects X and Y in $\mathcal{C}_T$, are $\mathcal{C}$ morphisms $X \to TY$.

That is we have a collection of objects A_T, one for each A in $\mathcal{C}$, and a collection of morphisms $f_T : A_T \to B_T$, each one corresponding to $f : A \to TB$. Given $f_T : A_T \to B_T$, and $g_T : B_T \to C_T$ we define the composition of these two morphism to be:

$$g_T \circ f_T = \mu_C \circ Tg_T \circ f_T$$

The identity morphism is taken to be the morphism $\eta_A : A \to TA$.

Proposition 0.5.14. *The Kleisli category is a category.*

Proof. We have a well defined collection of objects, and also we have a well defined collection of morphisms, each with a clear domain and codomain:

$$dom(f : A_T \to B_T) = A_T,\ cod(f : A_T \to B_T) = B_T.$$

The next property that we need to check is that composition is well defined. Given $f_T : A_T \to B_T$, and $g_T : B_T \to C_T$, we need to check that $g_T \circ f_T$, is a morphism in $\mathcal{C}_T$, that is we need to check that:

$$\mu_C \circ Tg \circ f.$$

Is a morphism of $\mathcal{C}$, mapping $A \to TC$. We know that $f : A \to TB$, and that $Tg : TB \to T^2C$, hence:

$$Tg \circ f : A \to T^2C$$

And as $\mu_C : T^2C \to TC$, we see that,

$$\mu_C \circ Tg \circ f : A \to TC.$$

And hence composition in $\mathcal{C}_T$, is well defined.

We now need to check that η_A is an identity morphism for the object A_T. We know that $\eta_A : A \to TA$, hence $\eta_T : A_T \to A_T$. Suppose we are given two morphisms $f_T : A_T \to B_T$, and $g_T : C_T \to A_T$, then we need to check that $(id_A)_T \circ g_T = g_T$, and $f_T \circ (id_A)_T = f_T$. That is:

$$\mu_A \circ T\eta_A \circ g = g.$$
$$\mu_B \circ Tf \circ \eta_A = f.$$

But we know that $\mu_A \circ T\eta_A = id_{TA}$, hence:

$$id_T A \circ g = g.$$
$$\mu_B \circ Tf \circ \eta_A = f.$$

So we see that the first condition is satisfied. To see the second condition, we first note that η is a natural transformation $id_{\mathcal{C}} \to T$, hence as $f : A \to TB$ is a morphism in $\mathcal{C}$, we must have the commuativity of the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ TB & \xrightarrow[\eta_{TB}]{} & T^2B \end{array} .$$

Therefore, $Tf \circ \eta_A = \eta_{TB} \circ f$. So the orginial requirement becomes:

$$\mu_B \circ (\eta_{BA} \circ f) = f.$$

And then we can once again use the conditions on a monad, that is $\mu_B \circ \eta_{BA} = id_{TB}$, hence:

$$id_{TB} \circ f = f.$$

Which we know to be true. Hence η_A, is an identity morphism.

The final thing we need to check is that composition is associative. Given $f_T : A_T \to B_T$, $g_T : B_T \to C_T$, and $h_T : C_T \to D_T$, we need to check that $(h_T \circ g_T) \circ f_T = h_T \circ (g_T \circ f_T)$. Which upon puting in the definition of composition in this category becomes:

$$(\mu_D \circ Th \circ g) \circ f_T = h_T \circ (\mu_C \circ Tg \circ f).$$

And then applying the definition of composition again tells us that we need:

$$\mu_D \circ T(\mu_D \circ Th \circ g) \circ f = \mu_D \circ Th \circ (\mu_C \circ Tg \circ f).$$

This identity is something that we wish to establish, so we can cancel the f off of both sides.

$$\mu_D \circ T(\mu_D \circ Th \circ g) = \mu_D \circ Th \circ \mu_C \circ Tg.$$

We can then expand out the T.

$$\mu_D \circ T\mu_D \circ T^2h \circ Tg = \mu_D \circ Th \circ \mu_C \circ Tg.$$

We know that μ is a natural transformation, hence we can take the mophism $h : C \to TD$, and form the following commutative diagram:

$$\begin{array}{ccc} T^2C & \xrightarrow{\mu_C} & TC \\ \downarrow{\scriptstyle T^3D} & & \downarrow{\scriptstyle Th} \\ T^3D & \xrightarrow[\mu_{TD}]{} & T^2D \end{array}$$

Therefore, $Th \circ \mu_D = \mu_{TD} \circ T^2h$, inserting this into the right hand side of the requirement we get that:

$$\mu_D \circ T\mu_D \circ T^2h \circ Tg = \mu_D \circ \mu_{TD} \circ T^2h \circ Tg.$$

Finally we can use the monad property that $\mu_D \circ \mu T_D = \mu_D \circ T\mu_D$ to reduce the requirment to:

$$\mu_D \circ \mu T_D \circ T^2h \circ Tg = \mu_D \circ \mu_{TD} \circ T^2h \circ Tg.$$

Hence we have shown that $\mathcal{C}_T$ is a category. $\square$

Now, in much the same way as the Eilenberg Maclane category, we wish to consturct an adjunction. We first define a functor $F : \mathcal{C} \to \mathcal{C}_T$, which sends:

$$FX = X \text{ for all objects } X \text{ in } \mathcal{C}.$$
$$F(f : X \to Y) = \eta_Y \circ f.$$

And another functor $U : \mathcal{C}_T \to \mathcal{C}$, which sends:

$$UY = TY \text{ for all objects } X \text{ in } \mathcal{C}.$$
$$U(g : X \to TY) = \mu_Y \circ Tf.$$

Then $F \dashv U$.
We need to check that both F and U are functors, and then we need to esablish the adjunction.

Proposition 0.5.15. *$F : \mathcal{C} \to \mathcal{C}_T$ is a functor.*

Proof. First note that F sends objects of $\mathcal{C}$ to objects of $\mathcal{C}_T$. Secondly note that $\eta_Y \circ f$ maps $X \to Y \to TY$, and that $\eta_Y \circ f$ is a morphism of $\mathcal{C}$, hence it is a morphism of $\mathcal{C}_T$, $X_T \to Y_T$. Now we need to check that F respects identities and composition. To check the identity requirement we need that for all $f_T : A_T \to B_T$, $g_T : C_T \to A_T$:

$$f_T \circ F(id_A) = f_T.$$
$$F(id_A) \circ g_T = g_T.$$

We know that, $F(id_A) = \eta_A \circ id_A$, hence putting in this definition, and using the definition of composition, we require:

$$\mu_B \circ Tf \circ (\eta_A \circ id_A) = f.$$
$$\mu_A \circ T\eta_A \circ (id_A \circ g) = g.$$

The second identity falls out easier, we simply note that from the definition of a monad, $\mu_A \circ T\eta_A = id_A$, hence the second requirement becomes:

$$id_A \circ (id_A \circ g) = g.$$

Which we know to be true. The first requirement is slightly harder, we need to recall that η is a natural transformation, hence we can form the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ TB & \xrightarrow[\eta_{TB}]{} & T^2B \end{array}$$

That is, $Tf \circ \eta_A = \eta_{TB} \circ f$. Putting this into the first requirement, we find that we need:

$$\mu_B \circ \eta_{TB} \circ f \circ id_A = f.$$

Then we can cancel the identity, and also insert the monad identity that, $\mu_B \circ \eta_{TB} = id_B$, giving:

$$id_B \circ f = f.$$

Which we know to be true. Hence F respects identities. Finally we need to check that F respects composition. That is, for all $f : A \to B$, $g : B \to C$, we need that:

$$F(g \circ f) = F(g) \circ F(f).$$

But, $F(g \circ f) = \eta_C \circ g \circ f$, $Fg = \eta_C \circ g$, $Ff = \eta_B \circ f$, hence we need:

$$\eta_C \circ g \circ f = (\eta_C \circ g) \circ (\eta_B \circ f).$$

Where the composition of the two brackets on the right is composition in $\mathcal{C}_T$, therefore, inserting the definition of composition:

$$\eta_C \circ g \circ f = \mu_C \circ T(\eta_C \circ g) \circ (\eta_B \circ f).$$

We can expand the T to get:

$$\eta_C \circ g \circ f = \mu_C \circ T\eta_C \circ Tg \circ (\eta_B \circ f).$$

But then from the monad property that $\mu_C \circ T\eta_C = id_C$, this reduces to:

$$\eta_C \circ g \circ f = Tg \circ (\eta_B \circ f).$$

We know that η is a natural transformation, hence as $g : B \to C$ is a morphism in $\mathcal{C}$:

$$\eta_C \circ g = Tg \circ \eta_B.$$

Inserting this we find that we need:

$$\eta_C \circ g \circ f = \eta_C \circ g \circ f.$$

Which holds, hence F is a functor. $\square$

Proposition 0.5.16. *U is a functor.*

Proof. First note that U sends objects of $\mathcal{C}$ to objects of $\mathcal{C}_T$, and sends morphism of $\mathcal{C}$ to morphisms of $\mathcal{C}_T$. Moreover it does so in the appropriate way, that is, $U(g : X_T \to y_T) : U(X_T) \to U(Y_T)$, but $U(X_T) = TX$, and $U(Y_T) = YT$, hence we want $Ug : TX \to YT$, but this is precisely what we have.

Finally we need to check that U respects identities and composition. Checking that U respects identities comes down to requireing that for all $f : TA \to B$, $g : C \to TA$:

$$U(id_{A_T}) \circ g = g$$
$$f \circ U(id_{A_T}) = f.$$

But we already know what id_{A_T} is, $id_{A_T} = \eta_A$, therefore we want:

$$U(\eta_A) \circ g = g$$
$$f \circ U(\eta_A) = f.$$

Then putting in the definition of U:

$$\mu_A \circ T\eta_A \circ g = g$$
$$f \circ \mu_A \circ T\eta_A = f.$$

Now we are in a position to use the monad identity: $\mu_A \circ T\eta_A = id_A$, giving us the required result. So we see that U respects identities. To finish the proof we now need to show that U respects composition. That is given $g_T : B_T \to C_T$, $f_T : A_T \to B_T$ we want that:

$$U(g_T \circ f_T) = Ug_T \circ Uf_T.$$

Then using the definition of composition in the category $\mathcal{C}_T$, we need:

$$U(\mu_C \circ Tg \circ f) = Ug_T \circ Uf_T.$$

Then putting in the definition of U, we get:

$$\mu_C \circ T(\mu_C \circ Tg \circ f) = (\mu_C \circ Tg) \circ (\mu_B \circ Tf).$$

Expanding the T:

$$\mu_C \circ T\mu_C \circ T^2 g \circ Tf = (\mu_C \circ Tg) \circ (\mu_B \circ Tf).$$

As μ is a natural transformation, and g is a morphism of $\mathcal{C}$, we have $Tg \circ \mu_B = \mu_{TC} \circ T^2 g$, hence:

$$\mu_C \circ T\mu_C \circ T^2 g \circ Tf = \mu_C \circ \mu_{TC} \circ T^2 g \circ Tf.$$

Then we can use the monad property that $\mu_C \circ T\mu_C = \mu_C \circ \mu_{TC}$, we need:

$$\mu_C \circ T\mu_C \circ T^2 g \circ Tf = \mu_C \circ T\mu_C \circ T^2 g \circ Tf.$$

So we see that U respects composition, and is therefore a functor. $\square$

Now that we have shown that F and U are functors, we are in a position to show that they are adjoint functors. We know that:

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C}_T$$

We wish to construt a natural isomorphism:

$$\phi : Hom(FX, Y_T) \to Hom(X, UY_T)$$

That is $\phi : Hom(X_T, Y_T) \to Hom(X, TY)$. However we already have just such a bijection, namely the bijection $(f_T : X_T \to Y_T) \mapsto (f : X \to TY)$. We now need to show that this bijection is natural in X and Y_T, that is for all $f : X_1 \to X_2$ in $\mathcal{C}$, $g_T : {Y^1}_T \to {Y^2}_T$ in $\mathcal{C}_T$. The following diagrams commute:

$$\begin{array}{ccc} Hom(FX, Y_T^1) & \xrightarrow{\phi_{X,Y^1}} & Hom(X, U(Y_T^1)) \\ \Big\downarrow {\scriptstyle Hom(FX,g_T)} & & \Big\downarrow {\scriptstyle Hom(X,Ug_T)} \\ Hom(FX, Y_T^2) & \xrightarrow[\phi_{X,Y^2}]{} & Hom(X, U(Y_T^1)) \end{array} \qquad \begin{array}{ccc} Hom(FX^2, Y_T) & \xrightarrow{\phi_{X^2,Y}} & Hom(X^2, U(Y_T)) \\ \Big\downarrow {\scriptstyle Hom(Ff,Y_T)} & & \Big\downarrow {\scriptstyle Hom(f,UY_T)} \\ Hom(FX^1, Y_T) & \xrightarrow[\phi_{X^2,Y}]{} & Hom(X^1, U(Y_T)) \end{array}$$

The first diagram requires that, for all $h_T : FX \to Y_T$:

$$Hom(X, Ug_T) \circ \phi_{X,Y^1}(h_T) = \phi_{X,Y^2} \circ Hom(FX, g_T)(h_T).$$

Putting in the definition of $Hom(A, -)$ gives us:

$$Ug_T \circ \phi_{X,Y^1}(h_T) = \phi_{X,Y^2} \circ (g_T \circ h_T).$$

Then putting in the definition of U, we get:

$$\mu_{Y^2} \circ Tg \circ \phi_{X,Y^1}(h_T) = \phi_{X,Y^2}(g_T \circ h_T).$$

If we let $\phi(h_t) = h$, then:

$$\mu_{Y^2} \circ Tg \circ h = \phi_{X,Y^2}(g_T \circ h_T).$$

Inserting the definition of composition in the category $\mathcal{C}_T$, gives us:

$$\mu_{Y^2} \circ Tg \circ h = \phi_{X,Y^2}((\mu_{Y^2} \circ Tg \circ h)_T).$$

And then applying ϕ on the right hand side:

$$\mu_{Y^2} \circ Tg \circ h = \mu_{Y^2} \circ Tg \circ h.$$

So we see that the first diagram commutes. The second diagram commutes when for all morphisms $i_T : FX^2 \to Y_T$:

$$Hom(f, UY_T) \circ \phi_{X^2,Y}(i_T) = \phi_{X^2,Y} \circ Hom(Ff, y_T)(i_T).$$

Putting in the definition of $Hom(-, A)$, tells us:

$$\phi_{X^2,Y}(i_T) \circ f = \phi_{X^2,Y}(i_T \circ Ff).$$

If we define $\phi_{X^2,Y}(i_T)$ to be i, then:

$$i \circ f = \phi_{X^2,Y}(i_T \circ Ff).$$

Putting in the definition of composition in $\mathcal{C}_T$, we get:

$$i \circ f = \phi_{X^2,Y}((\mu_Y \circ Ti \circ Ff)_T).$$

Inserting the definition of Ff,

$$i \circ f = \phi_{X^2,Y}((\mu_Y \circ Ti \circ (\eta_{X^2} \circ f)_T).$$

We know that η is a natural transformation hence, $Ti \circ \eta_{X^2} = \eta_{TY} \circ i$, therefore:

$$i \circ f = \phi_{X^2,Y}((\mu_Y \circ \eta_{TY} \circ i \circ f)_T).$$

And then as $\mu_Y \circ \eta_{TY} = id_Y$ from the monad property:

$$i \circ f = \phi_{X^2,Y}((i \circ f)_T).$$

And then applying ϕ on the right:

$$i \circ f = i \circ f.$$

Therefore we conclude that $F \dashv U$, and showing that we have constructed another adjunction from a given monad.

0.6 Conclusion

The aims of this project were as follows:
• To explore the idea that adjoint functors, and hence adjointness is pervasive in modern mathematics.
• To provide a path towards understanding adjointness which is as easy for the undergraduate mathematician to understand as possible.
• To give a grounding in the basics topics of Category Theory along the way.

If we are now to view the project from the point of view of these objectivies, then we see that almost everything covered up to adjoint functors was completely neccessary. The definitions of categories, functors and natural transformations were logically necessary in order to state the definition of adjoint functors. But moreover, some familiarity with them is required in order to grasp what is being given in the definition. This requirement justifies the development of limits and other constructions on categories.

However this isn't the only purpose that limits serve in this project, on the one hand they provide a means of furthering the understanding of categories, but they also provide an excellent example of adjoint functors. This is also the justification for the inclusion of the free constructions and the field of fractions. Once we introduced adjunction, we needed examples.

The rational for including monads was to provide a new perspective on adjunctions. We saw that every adjunction gives rise to a monad, and that every monad gives rise to at least two adjunctions, we therefore see that the concepts are intertwined. I would have liked to have spent more time exploring monads, but time and space constraints proved prohibitive.

I hope you enjoyed reading this project and that it gave some insight into the pervasiveness of adjoint functors.

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Section 2 is largely based on [4] and [3].
Section 3 is based on [2], though the limit theorem is modeled on a more basic version from [3].
Section 4 is also largely based on [2], though I also found [5] very useful. The field of fractions construction in section 4.1 was based on [6].
Section 5 mainly follows [5], though [2] was helpful at times.
I also consulted [1] throughout, especially for the section on comma categories.